Weight of closed subsets topologically generating a compact group*

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Dedicated to A.V. Arhangel'skiĭ on the occasion of his 66th anniversary

Abstract

Let G be a compact Hausdorff group. A subspace X of G *topologically generates* G if G is the smallest closed subgroup of G containing X. Define $tgw(G) = \omega \cdot \min\{w(X) : X$ is closed in G and topologically generates $G\}$, where $w(X)$ is the weight of X, i.e. the smallest size of a base for the topology of X. We prove that: (i) $tgw(G) = w(G)$ if G is totally disconnected, (ii) $tgw(G) = \sqrt[\omega]{w(G)} = \min\{\tau \geq \omega : w(G) \leq \tau^\omega\}$ in case G is connected, and (iii) $tgw(G) = w(G/c(G)) \cdot \sqrt[\omega]{w(c(G))}$, where $c(G)$ is the connected component of G. If G is connected, then either $tgw(G) = w(G)$, or $\mathrm{cf}(tgw(G)) = \omega$ (and, moreover, $w(G) = tgw(G)^+$ under the Singular Cardinal Hypothesis).

We also prove that $tgw(G) = \omega \cdot \min\{|X| : X \subseteq G$ is a compact Hausdorff space with at most one non-isolated point topologically generating $G\}$.

1 Introduction

All groups considered in this paper are assumed to be Hausdorff, and usually also compact. A subspace X of a topological group G *algebraically generates* G provided that the smallest subgroup $\langle X \rangle$ of G that contains X coincides with G, and we say that X *topologically generates* G if $\langle X \rangle$ is dense in G. Define

$$agw(G) = \min\{w(X) : X \text{ is closed in } G \text{ and algebraically generates } G\}$$

and

$$tgw(G) = \min\{w(F) : F \text{ is closed in } G \text{ and topologically generates } G\},$$

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where $w(X)$ is the weight of X, i.e. the smallest size of a base for the topology of X if such a base is infinite, or ω otherwise. We will call $agw(G)$ the *algebraically generating weight of* G and $tgw(G)$ the *topologically generating weight of* G. Clearly $tgw(G) \leq agw(G) \leq w(G)$. While the definition of algebraically generating weight appears to be more natural than that of topologically generating weight, it does not lead to anything new for compact groups: Arhangel'skiĭ proved that $agw(G) = w(G)$ for every compact group G [1, Proposition 3.8]. Since in our article we consider mostly compact groups, we will concentrate on the topologically generating weight, with the aim of computing completely the value $tgw(G)$ of topologically generating weight for a compact group G.

According to Hofmann and Morris [12], a subset X of a topological group G with the identity e is called *suitable* provided that:

(i) X topologically generates G,

(ii) $X \setminus \{e\}$ is a discrete subset of G, and

(iii) $X \cup \{e\}$ is a closed subset of G.

Suitable sets were considered first in the early sixties by Tate in the framework of Galois cohomology (see [7]). Tate proved[1] that every profinite group has a suitable set. Hofmann and Morris [12] proved that every (locally) compact group G has a suitable set and introduced the *generating rank*

$$s(G) = \min\{|X| : X \text{ is a suitable set of } G\}$$

of G. (For profinite groups, $s(G)$ has been already defined by Mel′nikov [17].) As witnessed by the fact that the whole Chapter 12 of the monograph [13] by Hofmann and Morris is devoted to the study of this cardinal function (and its relation to the weight), $s(G)$ is undoubtedly one of the most important cardinal invariants of a compact group G. This conclusion is further reinforced by the fact that Chapter 12 is the last chapter of the book [13] having its proofs built upon the whole power of various results and techniques from the preceding eleven chapters.

As an application of our results about $tgw(G)$, we establish that $tgw(G) = \omega \cdot s(G)$ for every compact group G, i.e. the topologically generating rank and the generating rank of a compact group coincide whenever the latter rank is infinite (Corollary 9.1). This equality makes all our results about $tgw(G)$ directly applicable to the study of $s(G)$ carried out by Hofmann and Morris in [12, 13]. We refer the reader to Section 9 for further details.

2 Overview of our principal results about $tgw(G)$

We start this section with pointing out that the obvious inequality

$$tgw(G) \leq w(G) \tag{1}$$

[1]This proof is extremely condensed. Detailed proofs can be found in [19] and [11].

holds for every group G. The relation between $tgw(G)$ and $w(G)$ becomes especially transparent in the totally disconnected case:

Theorem 2.1 *$tgw(G) = w(G)$ for a totally disconnected compact group G.*

As an easy corollary we obtain the well-known fact that every profinite topologically finitely generated group is metrizable.

In a particular case when the group G is Abelian, Theorem 2.1 is essentially due to Piskunov [18].

We note that Corollary 7.6(i) provides a slightly more general version of Theorem 2.1.

For an infinite cardinal τ define $\sqrt[\omega]{\tau}$ to be the smallest infinite cardinal κ with $\tau \leq \kappa^\omega$. Clearly, $\sqrt[\omega]{\tau} \leq \tau$. In the connected case the value of $tgw(G)$ can be expressed in terms of the "root function" $\sqrt[\omega]{-}$ defined above:

Theorem 2.2 *$tgw(G) = \sqrt[\omega]{w(G)}$ for every compact connected group G.*

Finally, we can combine the above two cases to get a general theorem:

Theorem 2.3 *$tgw(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G))$ for every compact group G, where $c(G)$ is the connected component of G.*

While $tgw(G)$ need not coincide with $w(G)$, the following double inequality can be proved:

Corollary 2.4 *$tgw(G) \leq w(G) \leq tgw(G)^\omega$ for every compact group G. In particular, $tgw(G)^\omega = w(G)^\omega$.*

Proof. The first inequality coincides with (1). To establish the second inequality, note that

$$w(G) = w(c(G)) \cdot w(G/c(G)) \leq \left(\sqrt[\omega]{w(c(G))}\right)^\omega \cdot w(G/c(G))^\omega \leq$$

$$\left(\sqrt[\omega]{w(c(G))}\right)^\omega \cdot w(G/c(G))^\omega = \left(\sqrt[\omega]{w(c(G))} \cdot w(G/c(G))\right)^\omega = tgw(G)^\omega,$$

where the last equality follows from Theorem 2.3. QED

Recall that the *cofinality* $\mathrm{cf}(\kappa)$ of an infinite cardinal κ is the smallest cardinal λ such that there exists a monotone function $f : \lambda \to \kappa$ with $\sup\{f(\alpha) : \alpha < \lambda\} = \kappa$.

For connected compact groups one can describe precisely when the chain of inequalities in the above corollary collapses into a "chain" of equalities:

Theorem 2.5 *Let G be a connected compact group. Then $tgw(G) = tgw(G)^\omega$ if and only if $\mathrm{cf}(tgw(G)) > \omega$.*

The following "dichotomy" holds for compact connected groups:

Corollary 2.6 *If G is a connected compact group, then either $tgw(G) = w(G)$, or $\mathrm{cf}(tgw(G)) = \omega$.*

Proof. If $tgw(G) \neq w(G)$, then $tgw(G) \neq tgw(G)^\omega$ by Corollary 2.4, and thus $\mathrm{cf}(tgw(G)) = \omega$ by Theorem 2.5. QED

As it turns out, both cases in the above theorem happen "cofinally often":

Example 2.7 *For every cardinal σ there exist successor (thus regular) cardinals $\tau > \sigma$ and $\kappa > \sigma$ such that $\mathrm{cf}(tgw(\mathbb{T}^\tau)) = \omega$ and $\mathrm{cf}(tgw(\mathbb{T}^\kappa)) > \omega$ and consequently, $tgw(\mathbb{T}^\kappa) = \kappa$ by Corollary 2.6. (Here $\mathbb{T}$ is the torus group.)*

Following van Douwen [8], we denote by (†) the following statement: If $\kappa \geq \mathfrak{c}$ is a cardinal with $\mathrm{cf}(\kappa) > \omega$, then $\kappa^\omega = \kappa$, where $\mathfrak{c}$ denotes the cardinality of the continuum. Under (†), a much stronger version of Corollary 2.4 can be proved:

Theorem 2.8 *Assume (†). Let G be a compact group such that $w(G) > \mathfrak{c}$. If $w(G/c(G)) < w(G) = \rho^+$, where ρ is a limit cardinal with $\mathrm{cf}(\rho) = \omega$, then $tgw(G) = \rho$, and thus $tgw(G)^+ = w(G)$. Otherwise, $tgw(G) = w(G)$.*

Under (†), Theorem 2.8 yields a precise rule for computing $tgw(G)$ in terms of $w(G)$ that substantially strengthens the conclusion of Corollary 2.6:

Corollary 2.9 *Assume (†). Let G be a connected compact group such that $w(G) > \mathfrak{c}$. If $w(G) = \rho^+$, where ρ is a limit cardinal with $\mathrm{cf}(\rho) = \omega$, then $tgw(G) = \rho$, and thus $tgw(G)^+ = w(G)$. Otherwise, $tgw(G) = w(G)$.*

It turns out that the conclusions of both Theorem 2.8 and Corollary 2.9 are equivalent to (†):

Theorem 2.10 *Suppose the following statement holds: "If G is a connected compact Abelian group such that $w(G) > \mathfrak{c}$, then either $tgw(G) = w(G)$, or $tgw(G)^+ = w(G)$." Then (†) also holds.*

Let GCH denote the Generalized Continuum Hypothesis saying that $2^\tau = \tau^+$ for all cardinals τ. One can easily see that GCH $\to$ (†) (see, for example, [14, Theorem 5.15(iii)]). It is well-known that GCH is consistent with the usual Zermelo-Fraenkel axioms ZFC of set theory [14]. Therefore, (†), and thus the conclusions of both Theorem 2.8 and Corollary 2.9, are consistent with ZFC.

Axiom (†) can be viewed as rather mild additional set-theoretic assumption beyond ZFC. Indeed, the negation of (†) is equiconsistent with the existence of large cardinals. Let SCH denote the so-called Singular Cardinal Hypothesis: For every singular cardinal κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. (Recall that an infinite cardinal κ is called *singular* if $\mathrm{cf}(\kappa) < \kappa$.) It is easy to see that SCH $\to$ (†) (see, for example, [14, Theorem 5.22(ii)(b)]). Therefore, $\neg$(†) $\to$ $\neg$SCH. Dodd and Jensen [5, 6] proved that the consistency of ZFC + $\neg$SCH implies the existence of an inner model with a measurable cardinal. Hence, to violate (†) one needs the strength of large cardinals. In particular, if the conclusion of either Theorem 2.8 or Corollary 2.9 fails in some model of ZFC, then there must exist a measurable cardinal (in an inner model). Finally, Magidor [16] proved that the consistency of the existence of a supercompact cardinal implies the consistency of ZFC + $\neg$(†).[2]

Proofs of all Theorems above and construction of Example 2.7 are presented in Section 6. Further results about $tgw(G)$ can be found in Sections 7 and 8, while some applications are gathered in Section 9. In particular, in Corollary 9.5 we apply our results on $tgw(G)$ to solve the problem of computation of the precise value of the generating rank of semidirect products raised by Hofmann and Morris in [12]. As a by-product, we also provide completely self-contained topological proofs of a series of highly non-trivial results from Chapter 12 of the monograph by Hofmann and Morris [13].

3 Folklore facts

For topological spaces X and Y we use $C(X,Y)$ to denote the family of all continuous maps from X to Y. No topology is assumed on $C(X,Y)$. The following two lemmas are probably a part of topological folklore.

Lemma 3.1 *Let X be a compact space and H a separable metric space. Then $|C(X,H)| \le w(X)^\omega$.*

Proof. Let $\mathcal{C}$ be a countable base for the topology of H, and let $\mathcal{B}$ be a base for the topology of X such that $|\mathcal{B}| \le w(X)$. Let $\mathcal{V}$ be the family of all open F_σ-subsets of X. By compactness of X, for every $V \in \mathcal{V}$ there exists a countable subset $\mathcal{B}_V$ of $\mathcal{B}$ such that $V = \bigcup \mathcal{B}_V$. Therefore, $|\mathcal{V}| \le |\mathcal{B}|^\omega \le w(X)^\omega$. Since H is a separable metric space, each $U \in \mathcal{C}$ is an open F_σ-subset of H. Therefore, if $U \in \mathcal{C}$ and $f \in C(X,H)$, then $f^{-1}(U) \in \mathcal{V}$ by continuity of f. Now notice that for every $f \in C(X,H)$ the family $\{f^{-1}(U) : U \in \mathcal{C}\} \subseteq \mathcal{V}^{\mathcal{C}}$ uniquely determines f. This implies that $|C(X,H)| \le |\mathcal{V}^{\mathcal{C}}| \le |\mathcal{V}|^\omega \le (w(X)^\omega)^\omega = w(X)^\omega$. QED

Lemma 3.2 *Let X be a totally disconnected compact space and H be a discrete space. Then $|C(X,H)| \le w(X) \cdot |H|$.*

[2] Assuming the existence of a supercompact cardinal, Magidor [16] produced a model M of ZFC in which $2^{\omega_n} = \omega_{n+1}$ for every natural number n, while $2^{\omega_\omega} = \omega_{\omega+2}$. To show that $\neg$(†) holds in M, it suffices to note that $\kappa = \omega_{\omega+1}$ violates (†). Indeed, since $2^{\omega_n} = \omega_{n+1}$ for every natural number n, we must have $2^{\omega_\omega} = (\omega_\omega)^\omega$, which implies

$$\kappa^\omega = (\omega_{\omega+1})^\omega \ge (\omega_\omega)^\omega = 2^{\omega_\omega} = \omega_{\omega+2} = \kappa^+ > \kappa.$$

Proof. Since X is compact and totally disconnected, it is zero-dimensional, i.e. the family $\mathcal{E}$ of all clopen (=both open and closed) subsets of X is a base for its topology [9, Theorem 6.2.9]. Let $\mathcal{B}$ be a base for the topology of X such that $|\mathcal{B}| \leq w(X)$. Since each $E \in \mathcal{E}$ is a clopen subset of X, $E = \bigcup \mathcal{B}_E$ for some $\mathcal{B}_E \in [\mathcal{B}]^{<\omega}$ by compactness of E. (As usual, $[Z]^{<\omega}$ denotes the family of all finite subsets of a set Z.) Therefore, $|\mathcal{E}| \leq |[\mathcal{B}]^{<\omega}| \leq \omega \cdot |\mathcal{B}| \leq w(X)$.

Let $f \in C(X, H)$. Then $f(X)$ is a compact subset of the discrete group H, and so $f(X)$ is finite. Therefore, $\{f^{-1}(h) : h \in H\} \in [\mathcal{E}]^{<\omega}$. This defines an injection from $C(X, H)$ into $[\mathcal{E}]^{<\omega} \times [H]^{<\omega}$. Therefore, $|C(X, H)| \leq |[\mathcal{E}]^{<\omega}| \cdot |[H]^{<\omega}| \leq \omega \cdot |\mathcal{E}| \cdot |H| \leq w(X) \cdot |H|$. QED

The following result about compact connected groups is perhaps a part of folklore:

Theorem 3.3 (Universal compact connected group of a given weight) *There exists a sequence $\{L_n : n \in \mathbb{N}\}$ of compact connected simple Lie groups L_n such that every compact connected group of weight $\leq \tau$ is a quotient group of the group $G_\tau = \widehat{\mathbb{Q}}^\tau \times \prod_{n \in \mathbb{N}} L_n^\tau$, where $\widehat{\mathbb{Q}}$ is the Pontryagin dual of the discrete group $\mathbb{Q}$ of rational numbers.* (Note that G_τ is a connected group of weight τ.)

Proof. It follows from the Varopoulos Theorem [21] that a connected compact group G of weight τ is a quotient group of the direct product $A \times L$, where $L = \prod_{n \in \mathbb{N}} L_n^\tau$ is as in the conclusion of our theorem and A is the connected component of the center of G. Therefore, A is a connected compact Abelian group with $w(A) \leq w(G) = \tau$, and so the Pontryagin dual $\widehat{A}$ of A is a discrete torsion-free Abelian group of cardinality $\leq \tau$. Then $\widehat{A}$ is contained in a torsion-free divisible group H of cardinality τ. Since H is a linear space over $\mathbb{Q}$, H is algebraically isomorphic to the direct sum of τ many copies of $\mathbb{Q}$, and so $\widehat{H} \cong \widehat{\mathbb{Q}}^\tau$. Taking the Pontryagin duals again, we get a continuous surjective homomorphism $\widehat{H} \cong \widehat{\mathbb{Q}}^\tau \to A$ that must be a quotient homomorphism as the groups are compact. Therefore, G is a quotient group of $\widehat{\mathbb{Q}}^\tau \times \prod_{n \in \mathbb{N}} L_n^\tau$. QED

4 Facts about super-sequences, $tgw(G)$ and $seq(G)$

Definition 4.1 A *super-sequence* is a non-empty compact Hausdorff space with at most one non-isolated point. The non-isolated point x^* of a super-sequence X, if exists, will be called the *limit* of X, and we will say that X *converges to* x^*.

The choice of the name is motivated by the fact that countable super-sequences are precisely the usual convergent sequences, with the usual meaning of limit and convergence.

Fact 4.2 *A Hausdorff space X is a super-sequence if and only if there exists a point $x^* \in X$ such that the set $X \setminus U$ is finite for every open set $U \subseteq X$ containing x^*. Moreover, if a super-sequence X has a limit, then this limit must coincide with x^* above.*

Our previous fact immediately implies the following

Fact 4.3 *A continuous image of a super-sequence is again a super-sequence.*

Fact 4.4 *If X is a super-sequence topologically generating a topological group G, and x^* is the limit of X, then $S = \{x^*\} \cup (x^*)^{-1} \cdot X$ is a super-sequence topologically generating G and converging to $e \in S$. Moreover, $|S| \leq |X| + 1$.*

Definition 4.5 Let $\{G_i : i \in I\}$ be a family of topological groups. For every $i \in I$, let S_i be a super-sequence in G_i such that S_i converges to the identity element e_i of G_i whenever S_i is infinite. Identifying each G_i with a subgroup of the direct product $G = \prod_{i \in I} G_i$ in the obvious way, define $S = \bigcup_{i \in I} S_i \cup \{e\}$, where e is the identity element of G. We will call S the *fan of the super-sequences* $\{S_i : i \in I\}$ and will denote it by $\mathrm{fan}_{i \in I}(S_i, G_i)$. If $G_i = G$ and $S_i = S$ are constant, we write simply $\mathrm{fan}_I(S, G)$.

Lemma 4.6 *Let $\{G_i : i \in I\}$ and $\{S_i : i \in I\}$ be as in Definition 4.5. Then:*
(a) $|\mathrm{fan}_G(\{S_i : i \in I\})| \leq \sup\{|S_i| : i \in I\} + |I| + \omega$,
(b) $\mathrm{fan}_{i \in I}(S_i, G_i)$ *is a super-sequence in* $G = \prod_{i \in I} G_i$*, and*
(c) if each S_i topologically generates G_i, then $\mathrm{fan}_{i \in I}(S_i, G_i)$ *topologically generates G.*

Proof. (a) is obvious.
(b) Follows directly from Fact 4.2, since the set $S \setminus U$ is finite for every open set $U \subseteq G$ containing e.
(c) S topologically generates G since the closed subgroup generated by $\mathrm{fan}_{i \in I}(S_i, G_i)$ contains the direct sum $\bigoplus_{i \in I} G_i$ of the groups G_i, and this direct sum is dense in G. QED

Definition 4.7 For a topological group G we define

$$seq(G) = \omega \cdot \min\{|S| : S \subseteq G \text{ is a super-sequence topologically generating } G\}$$

provided that G has at least one super-sequence topologically generating it, and we consider $seq(G)$ to be undefined otherwise.

We also agree that, whenever $seq(G)$ appears in print, it implicitly means that $seq(G)$ is defined.

Lemma 4.8 $seq\left(\prod_{n \in \mathbb{N}} G_n\right) \leq \sup\{seq(G_n) : n \in \mathbb{N}\}$.

Proof. Let $\tau = \sup\{seq(G_n) : n \in \mathbb{N}\}$ and $G = \prod_{n \in \mathbb{N}} G_n$. Let S_n be a super-sequence in G_n topologically generating G_n with $|S_n| \leq seq(G_n) \leq \tau$. Since τ is infinite, Fact 4.4 allows us to assume, without loss of generality, that the limit point of S_n, if exists, coincides with e_n, the identity element of G_n. According to Lemma 4.6, the super-sequence $S = \mathrm{fan}_{n \in \mathbb{N}}(S_n, G_n)$ in G topologically generates G and $|S| \leq \tau$. Hence $seq(G) \leq \omega \cdot |S| \leq \tau$. QED

In the sequel we prove that equality holds in the lemma (see Theorem 7.1 combined with Theorem 8.1).

Fact 4.9 *If H is a quotient group of a compact group G, then $tgw(H) \leq tgw(G)$.*

Proof. Let $\pi : G \to H$ be the quotient map and let X be a closed subset of G topologically generating G such that $w(X) = tgw(G)$. Since G is compact, so are X and $Y = \pi(X)$. In particular, Y is closed in H. Since Y is a continuous image of X, it follows that $w(Y) \leq w(X)$ (see, for example, [9, Theorem 3.1.22]). Since X topologically generates G, one concludes that Y topologically generates H. Therefore, $tgw(H) \leq w(Y) \leq w(X) = tgw(G)$. QED

Lemma 4.10 *If H is a closed normal subgroup of a compact group G, then $tgw(G) \leq tgw(H) \cdot w(G/H)$.*

Proof. Let $\pi : G \to G/H$ be the quotient homomorphism. According to results of Hofmann and Morris [12] (see also Section 1), G has a suitable set X. Since G is compact, X must be a super-sequence X topologically generating G. Since the image $\pi(X)$ of X is a super-sequence in G/H (Fact 4.3), it follows that $|\pi(X)| \leq w(\pi(X)) \leq w(G/H)$. For every $y \in \pi(X) \setminus \{\pi(e)\}$ choose $x_y \in X$ such that $\pi(x_y) = y$. Then $X' = \{x_y : y \in \pi(X) \setminus \{\pi(e)\}\} \cup \{e\}$ is a super-sequence such that $\pi(X') = \pi(X)$. Since X topologically generates G, $\pi(X') = \pi(X)$ topologically generates G/H. Clearly $|X'| \leq \omega \cdot |\pi(X)| \leq w(G/H)$. Let K be a compact set in H such that $w(K) = tgw(H)$ and K topologically generates H. Now $X' \cup K$ is a compact (and thus closed) set in G. Since $w(X' \cup K) \leq tgw(H) \cdot w(G/H)$, it remains only to prove that $X' \cup K$ topologically generates G.

Let N be a closed subgroup of G that contains $X' \cup K$. Since N is compact, $\pi(N)$ is a compact subgroup of G/H. In particular, $\pi(N)$ is closed in G/H. Since $\pi(X') \subseteq \pi(N)$ and $\pi(X')$ topologically generates G/H, we conclude that $\pi(N) = G/H$. Since K topologically generates H and $K \subseteq N$, we conclude that $H \subseteq N$. From both $\pi(N) = G/H$ and $H \subseteq N$ it now follows that $G \subseteq N$. QED

5 Topologically finitely generated groups without proper open subgroups

We say that a topological group G is *topologically finitely generated* if G has a finite subset topologically generating it.

Lemma 5.1 *For a topological group G the following conditions are equivalent:*

(a) G is topologically finitely generated and has no proper open subgroups,

(b) every non-empty open set $U \subseteq G$ contains a finite subset topologically generating G.

Proof. Obviously, (b) implies (a). Let us prove that (a) implies (b). Assume that G is a topologically finitely generated group without proper open subgroups. Let X be a finite subset of G topologically generating G; that is, the subgroup $N = \langle X \rangle$ of G is dense in G.

First we shall prove a special case of our item (b) when $e \in U$.The subgroup $H = \langle N \cap U \rangle$ of N is obviously open in N, hence also closed in N. Since U is open in G and N is dense in G, we have $\emptyset \neq U \subseteq \overline{U} = \overline{N \cap U} \subseteq \overline{\langle N \cap U \rangle} = \overline{H}$, where the bar denotes the closure in G. Therefore, $\overline{H}$ is an open subgroup of G, and our hypothesis gives $\overline{H} = G$. Now closedness of H in N yields $H = \overline{H} \cap N = G \cap N = N$. We have proved in this way that $X \subseteq N = H = \langle N \cap U \rangle$. In particular, for each $x \in X$ there exists a finite set $F_x \subseteq N \cap U$ such that $x \in \langle F_x \rangle$. Clearly, the set $F = \bigcup_{x \in X} F_x \subseteq U$ is finite and $N = \langle X \rangle \subseteq \langle F \rangle$. Since N is dense in G, it follows that F topologically generates G.

Now we shall consider the general case. Let $U \neq \emptyset$ be an arbitrary open subset of G. Then $V = UU^{-1}$ is an open subset of G with $e \in V$, and the special case of item (b) proved above implies the existence of a finite set $\Phi \subseteq V$ topologically generating G. For each $g \in \Phi$ one can find $y_g, z_g \in U$ such that $g = y_g z_g^{-1}$. Now $F = \bigcup_{g \in \Phi} \{y_g, z_g\}$ is a finite subset of U such that $\langle F \rangle \supseteq \langle \Phi \rangle$. Since the latter set is dense in G, F topologically generates G. QED

Theorem 5.2 *Let G be a metric topologically finitely generated group without proper open subgroups. Then $seq(G^{\tau^\omega}) \leq \tau$ for every infinite cardinal τ.*

Proof. Fix an infinite cardinal τ. We agree to consider τ^ω as the set of all functions $f : \omega \to \tau$ from ω to τ. For $f \in \tau^\omega$ and $n \in \omega$ let $f \restriction n \in \tau^n$ be the restriction of the function f to the set $n = \{0, 1, \dots, n-1\}$. For every finite subset F of τ^ω denote by π_F the canonical projection $G^{\tau^\omega} \to G^F$. If $F = \{f\}$ is a singleton, write simply π_f instead of $\pi_{\{f\}}$.

Fix a decreasing open base $\{U_n : n \in \omega\}$ at the identity element e of G such that $U_0 = G$. For $g \in G \setminus \{e\}$ the natural number

$$v(g) = \max\{n \in \mathbb{N} : g \in U_n\} \tag{2}$$

is well-defined, so for each $h \in \tau^\omega$ we can define a function $x_{g,h} \in G^{\tau^\omega}$ as follows:

$$x_{g,h}(f) = \begin{cases} g, & \text{if } f \restriction v(g) = h \restriction v(g), \\ e, & \text{if } f \restriction v(g) \neq h \restriction v(g) \end{cases} \qquad \text{for all } f \in \tau^\omega. \tag{3}$$

Claim 5.2.1 *If $g \in G \setminus \{e\}$, then:*
(i) $x_{g,h} = x_{g,h'}$ whenever $h, h' \in \tau^\omega$ and $h \restriction v(g) = h' \restriction v(g)$,
(ii) $x_{g,h}(f) \in U_{v(g)}$ for all $f \in \tau^\omega$,
(iii) $|\{x_{g,h} : h \in \tau^\omega\}| \leq \tau$.

Proof. Note that $g \in U_{v(g)}$ by (2), so items (i) and (ii) follow immediately from (3). Furthermore, (i) implies $|\{x_{g,h} : h \in \tau^\omega\}| \leq \tau^{v(g)} = \tau$. QED

For each $n \in \omega$ use Lemma 5.1 to fix a finite set $S_n \subseteq U_n$ topologically generating G. Without loss of generality, we will assume that $e \notin S_n$. Define $S = \bigcup_{n=1}^{\infty} S_n$ and note that $S \cup \{e\}$ is a (super)sequence in G converging to e.

Define $X = \{x_{g,h} : g \in S, h \in \tau^\omega\}$. Since S is countable, Claim 5.2.1(iii) yields

Claim 5.2.2 $|X| \leq \tau$.

Claim 5.2.3 *$X \setminus \pi_f^{-1}(U_k)$ is finite whenever $f \in \tau^\omega$ and $k \in \omega$.*

Proof. Fix $f \in \tau^\omega$ and $k \in \omega$. Since the set $S \setminus U_k$ is finite, it suffices to prove that $X \setminus \pi_f^{-1}(U_k) \subseteq \{x_{g,f} : g \in S \setminus U_k\}$. Suppose that $g \in S$, $h \in \tau^\omega$ and $x_{g,h} \notin \pi_f^{-1}(U_k)$. Then $x_{g,h}(f) = \pi_f(x_{g,h}) \notin U_k$. If $k \leq v(g)$, then $x_{g,h}(f) \in U_{v(g)} \subseteq U_k$ by Claim 5.2.1(ii) and monotonicity of the sequence $\{U_n : n \in \omega\}$, which gives a contradiction. Therefore, we must have $v(g) < k$, which yields $g \notin U_k$ by (2). We have proved that $g \in S \setminus U_k$. Finally, note that $f \restriction v(g) \neq h \restriction v(g)$ implies $x_{g,h}(f) = e \in U_k$ by (3), which again gives a contradiction. Hence, we must have $f \restriction v(g) = h \restriction v(g)$, which yields $x_{g,f} = x_{g,h}$ by Claim 5.2.1(i). QED

Claim 5.2.4 *$X \cup \{1\}$ is a super-sequence, where 1 is the identity element of the group G^{τ^ω}.*

Proof. According to Fact 4.2, it suffices to prove that the set $X \setminus W$ is finite for every open subset W of G^{τ^ω} containing 1. Since $\{U_n : n \in \omega\}$ is a decreasing open base at e, there exist $k \in \omega$ and a finite set $F \subseteq \tau^\omega$ such that $\bigcap_{f \in F} \pi_f^{-1}(U_k) \subseteq W$. Now note that

$$X \setminus W \subseteq X \setminus \bigcap_{f \in F} \pi_f^{-1}(U_k) = \bigcup_{f \in F} X \setminus \pi_f^{-1}(U_k),$$

and the latter set is finite by Claim 5.2.3. QED

Claim 5.2.5 *$\pi_F(X)$ topologically generates G^F for every finite subset F of τ^ω.*

Proof. Fix an $n \in \omega$ such that $f' \restriction n \neq f'' \restriction n$ whenever $f', f'' \in F$ and $f' \neq f''$. Since S_n topologically generates G, $\mathrm{fan}_F(S_n, G)$ topologically generates G^F by Lemma 4.6. Therefore, it suffices to prove that $\mathrm{fan}_F(S_n, G) \setminus \{e_F\} \subseteq \pi_F(X)$, where e_F is the identity element of G^F.

Indeed, pick $y \in \mathrm{fan}_F(S_n, G) \setminus \{e_F\}$ arbitrarily. By Definition 4.5 there exist $h \in F$ and $g \in S_n$ such that $y(h) = g$ and $y(f) = e$ for all $f \in F \setminus \{h\}$. Note that $x_{g,h}(h) = g = y(h)$ by (3). Let $f \in F \setminus \{h\}$. Since $g \in S_n \subseteq U_n$, we have $n \leq v(g)$ by (2), and so $f \restriction n \neq h \restriction n$ implies $f \restriction v(g) \neq h \restriction v(g)$. Now (3) yields $x_{g,h}(f) = e = y(f)$. We have proved that $y = \pi_F(x_{g,h}) \in \pi_F(X)$. QED

Claim 5.2.6 *$X \cup \{1\}$ topologically generates G^{τ^ω}.*

Proof. We need to prove that the subgroup $H = \langle X \rangle$ of G^{τ^ω} is dense in G^{τ^ω}. Let W be an arbitrary non-empty open subset of G^{τ^ω}. Then there exist a finite subset F of τ^ω and a non-empty open subset V of G^F such that $\pi_F^{-1}(V) \subseteq W$. By Claim 5.2.5, $\langle \pi_F(X) \rangle = \pi_F(\langle X \rangle) = \pi_F(H)$ is dense in G^F, so $\pi_F(H) \cap V \neq \emptyset$. Therefore, $\emptyset \neq H \cap \pi_F^{-1}(V) \subseteq H \cap W$. QED

Finally, from Claims 5.2.2, 5.2.4 and 5.2.6 one gets $seq(G^{\tau^\omega}) \leq \tau$. QED

Since connected groups do not have proper open subgroups, our next corollary immediately follows from Theorem 5.2.

Corollary 5.3 *Let G be a connected metric topologically finitely generated group. Then $seq(G^{\tau^\omega}) \leq \tau$ for every infinite cardinal τ.*

6 Computation of $tgw(G)$

For topological groups G and H we will use $\mathrm{Hom}(G, H)$ to denote the family of all continuous homomorphisms from G to H. No topology is assumed on $\mathrm{Hom}(G, H)$.

Lemma 6.1 *Let X be a subset of a topological group G. Assume that X topologically generates G. Then $|\mathrm{Hom}(G, H)| \leq |C(X, H)|$ for every topological group H.*

Proof. Define a map $f : \mathrm{Hom}(G, H) \to C(X, H)$ by $f(\pi) = \pi|_X$ for $\pi \in \mathrm{Hom}(G, H)$. We claim that f is an injection. Indeed, assume that $\pi, \varpi \in \mathrm{Hom}(G, H)$ and $f(\pi) = f(\varpi)$. Then $\pi|_X = \varpi|_X$. Since both π and ϖ are group homomorphisms from G to H, one has $\pi|_{\langle X \rangle} = \varpi|_{\langle X \rangle}$, where $\langle X \rangle$ is the smallest subgroup of G that contains X. Since $\langle X \rangle$ is dense in G, continuity of π and ϖ implies now that $\pi = \varpi$. QED

Lemma 6.2 *$tgw(G) \geq w(G)$ for a compact totally disconnected group G.*

Proof. Let X be a closed subset of G that topologically generates G. Since G is compact and totally disconnected, it is profinite, i.e. its topology is determined by the family of all continuous homomorphisms into finite discrete groups [10]. Let H be one of these discrete groups. Since G is totally disconnected, so is X. Therefore $|C(X, H)| \leq w(X)$ by Lemma 3.2. In view of Lemma 6.1, we also have $|\mathrm{Hom}(G, H)| \leq |C(X, H)|$. Since there are only countably many pairwise non-isomorphic finite discrete groups, it now follows that $w(G) \leq \omega \cdot w(X) = w(X)$. This proves $w(G) \leq tgw(G)$. QED

Proof of Theorem 2.1: By Lemma 6.3 $tgw(G) \geq w(G)$ for a totally disconnected compact group G. Now (1) finishes the proof.

Lemma 6.3 *$tgw(G) \geq w(G/c(G))$ for a compact group G.*

Proof. Since $G/c(G)$ is totally disconnected, Fact 4.9 and Lemma 6.2 imply $tgw(G) \geq tgw(G/c(G)) \geq w(G/c(G))$. QED

Theorem 6.4 *$tgw(G) \geq \sqrt[\omega]{w(G)}$ for every compact group G.*

Proof. Let G be a compact group topologically generated by its closed subset X with $w(X) = tgw(G)$. By Peter-Weyl-van Kampen theorem [10] the topology of G is determined by the set of its homomorphisms into the compact metric group $H = \prod_{n\in\mathbb{N}} \mathbb{U}(n)$, where $\mathbb{U}(n)$ is the group of unitary $n \times n$ matrices over the complex number field. Combining Lemmas 6.1 and 3.1 we obtain $|\mathrm{Hom}(G, H)| \leq |C(X, H)| \leq w(X)^\omega = tgw(G)^\omega$. This permits to conclude that $w(G) \leq tgw(G)^\omega$. Therefore $\sqrt[\omega]{w(G)} \leq tgw(G)$. QED

Lemma 6.5 *$tgw(G) \leq \sqrt[\omega]{w(G)}$ for a compact connected group G.*

Proof. Let $\tau = \sqrt[\omega]{w(G)}$. By Theorem 3.3, G is a quotient group of the group $H = \widehat{\mathbb{Q}}^{w(G)} \times \prod_{n\in\mathbb{N}} L_n^{w(G)}$ for a suitable sequence $\{L_n : n \in \mathbb{N}\}$ of compact connected simple Lie groups L_n. Since $w(G) \leq \tau^\omega$, H is a natural quotient group (under the projection map) of the group K^{τ^ω}, where $K = \widehat{\mathbb{Q}} \times \prod_{n\in\mathbb{N}} L_n$. Therefore $tgw(G) \leq tgw(H) \leq tgw(K^{\tau^\omega})$ by Fact 4.9. It remains only to prove that $tgw(K^{\tau^\omega}) \leq \tau$.

Since $\widehat{\mathbb{Q}}$ is monothetic, and so topologically finitely generated, we have $seq(\widehat{\mathbb{Q}}^{\tau^\omega}) \leq \tau$ by Corollary 5.3. According to the classical result of Kuranishi [15], each L_n has a dense subgroup generated by two elements, so again $seq(L_n^{\tau^\omega}) \leq \tau$ by Corollary 5.3. Now Lemma 4.8 allows us to conclude that $seq\left(\widehat{\mathbb{Q}}^{\tau^\omega} \times \prod_{n\in\mathbb{N}} L_n^{\tau^\omega}\right) \leq \tau$. Since K^{τ^ω} is (topologically) isomorphic to $\widehat{\mathbb{Q}}^{\tau^\omega} \times \prod_{n\in\mathbb{N}} L_n^{\tau^\omega}$, we get $tgw(K^{\tau^\omega}) \leq seq(K^{\tau^\omega}) \leq \tau$. QED

Proof of Theorem 2.2: Combine Theorem 6.4 and Lemma 6.5.

Proof of Theorem 2.3: Let G be a compact group. By Lemmas 4.10 and 6.5 we have

$$tgw(G) \leq tgw(c(G)) \cdot w(G/c(G)) \leq \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)).$$

On the other hand, Theorem 6.4 implies $tgw(G) \geq \sqrt[\omega]{w(G)} \geq \sqrt[\omega]{w(c(G))}$ (since the function $\sqrt[\omega]{-}$ is monotone). Combining this with Lemma 6.3, one concludes that

$$\sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) \leq tgw(G).$$

Fact 6.6 [14, Theorem 5.20(iii)(a)] *Suppose that κ is a cardinal, $\mathrm{cf}(\kappa) > \omega$ and $\tau^\omega \leq \kappa$ for each cardinal $\tau < \kappa$. Then $\kappa^\omega = \kappa$.*

Lemma 6.7 *If τ is a cardinal such that* $\mathrm{cf}\,(\sqrt[\omega]{\tau}) > \omega$*, then* $\sqrt[\omega]{\tau} = \tau = \tau^\omega$*.*

Proof. Define $\kappa = \sqrt[\omega]{\tau}$. Let $\alpha < \kappa$ be an arbitrary ordinal. If $|\alpha|^\omega \geq \kappa$, then $|\alpha|^\omega \geq \kappa^\omega \geq \tau$ because $\kappa = \sqrt[\omega]{\tau}$. Therefore, $\kappa \leq |\alpha|$, in contradiction with $|\alpha| < \kappa$. We have proved that $|\alpha|^\omega < \kappa$ for every ordinal $\alpha < \kappa$. Now $\kappa^\omega \leq \kappa$ by Fact 6.6. Since $\tau \leq \kappa^\omega$ by the definition of $\sqrt[\omega]{\tau}$, we get $\tau \leq \tau^\omega \leq \kappa^\omega \leq \kappa \leq \tau$. This proves $\kappa = \tau = \tau^\omega$. QED

PROOF OF THEOREM 2.5: Let $\tau = w(G)$ and $\kappa = tgw(G)$. Then $\kappa = \sqrt[\omega]{\tau}$ by Theorem 2.2. If $\mathrm{cf}(\kappa) = \omega$, then $\kappa < \kappa^\omega$ [14, Corollary 5.14]. If $\mathrm{cf}(\kappa) > \omega$, then $\kappa = \tau = \tau^\omega$ by Lemma 6.7, which yields $\kappa^\omega = \kappa$.

Lemma 6.8 *For every infinite cardinal σ there exist successor (thus regular) cardinals $\tau \geq \sigma$ and $\kappa \geq \sigma$ such that* $\mathrm{cf}(\sqrt[\omega]{\tau}) = \omega$ *and* $\mathrm{cf}(\sqrt[\omega]{\kappa}) > \omega$*.*

Proof. Let $\lambda > \sigma$ be any cardinal with $\mathrm{cf}(\lambda) = \omega$. Define $\tau = \lambda^+$ and $\kappa = (\lambda^\omega)^+$. Then $\lambda^\omega > \lambda$ [14, Corollary 5.14], and thus $\lambda^\omega \geq \tau$. It follows that $\sqrt[\omega]{\tau} \leq \lambda < \tau \leq (\sqrt[\omega]{\tau})^\omega$. Thus $\mathrm{cf}(\sqrt[\omega]{\tau}) = \omega$ by Lemma 6.7. Finally, note that $\sqrt[\omega]{\kappa} = \kappa$, and so $\mathrm{cf}(\sqrt[\omega]{\kappa}) = \mathrm{cf}(\kappa) > \omega$ as κ is a regular cardinal. QED

CONSTRUCTION OF EXAMPLE 2.7: Combine Lemma 6.8 and Theorem 2.2.

PROOF OF THEOREM 2.8: Let G be a compact group with $w(G) > \mathfrak{c}$.

(i) Assume first that $w(G/c(G)) < w(G) = \rho^+$, where ρ is a limit cardinal with $\mathrm{cf}(\rho) = \omega$. Clearly, $w(G/c(G)) \leq \rho$.

Let us prove that $\sqrt[\omega]{\rho^+} = \rho$. Since $\mathrm{cf}(\rho) = \omega$, we have $\rho^\omega > \rho$ by [14, Corollary 5.14]. Hence, $\sqrt[\omega]{\rho^+} \leq \rho$. To prove the reverse inequality, let $\kappa < \rho$ be an arbitrary cardinal. Since κ^+ is a regular cardinal, $\mathrm{cf}(\kappa^+) = \kappa^+ > \omega$, and so $(\kappa^+)^\omega = \kappa^+$ by (†). Therefore, $\kappa^\omega \leq (\kappa^+)^\omega = \kappa^+ \leq \rho < \rho^+$. This implies $\sqrt[\omega]{\rho^+} \geq \rho$.

Since $w(G/c(G)) < w(G)$ and $w(G) = w(c(G)) \cdot w(G/c(G))$, one has $w(G) = w(c(G))$, and so $\sqrt[\omega]{w(c(G))} = \sqrt[\omega]{w(G)} = \sqrt[\omega]{\rho^+} = \rho$. Now from Theorem 2.3 and $w(G/c(G)) \leq \rho$, we conclude that $tgw(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) = \rho$.

(ii) Suppose now that $tgw(G) < w(G)$. Define $\rho = tgw(G)$. Corollary 2.4 yields $\rho \leq w(G) \leq \rho^\omega$. Since $w(G) > \mathfrak{c}$, we must have $\rho > \mathfrak{c}$. If $\mathrm{cf}(\rho) > \omega$, then $\rho^\omega = \rho$ by (†), which implies $\rho = w(G)$, a contradiction. Therefore, $\mathrm{cf}(\rho) = \omega$, and thus $\rho^\omega \geq \rho^+$ by [14, Corollary 5.14]. Note that ρ^+ is a regular cardinal, so $\mathrm{cf}(\rho^+) = \rho^+ > \omega$. Since $\mathfrak{c} < \rho < \rho^+$, we have $\rho^\omega \leq (\rho^+)^\omega = \rho^+$ by (†). Therefore, $w(G) \leq \rho^\omega = \rho^+$. Since $\rho < w(G)$, it follows that $w(G) = \rho^+$.

PROOF OF THEOREM 2.10: Assume that (†) fails. Then there exists a cardinal $\kappa \geq \mathfrak{c}$ such that $\mathrm{cf}(\kappa) > \omega$ and $\kappa^\omega > \kappa$. Applying Fact 6.6, we can find a cardinal $\tau < \kappa$ such that $\tau^\omega > \kappa$. Note that $G = \mathbb{T}^{\tau^\omega}$ is a compact connected Abelian group with $w(G) = \tau^\omega$, so $tgw(G) = \sqrt[\omega]{\tau^\omega} \leq \tau$ by Theorem 2.2. Since $tgw(G)^+ \leq \tau^+ \leq \kappa < \tau^\omega = w(G)$, neither $tgw(G) = w(G)$, nor $tgw(G)^+ = w(G)$ holds.

7 Categorical properties of $tgw(G)$

7.1 Direct and semidirect products

In this section we collect some natural properties of $tgw(G)$ concerning products which follow easily from our explicit formulae. Our first result gives a rule for computing the topologically generating weight of arbitrary direct products of compact groups:

Theorem 7.1 *Let $\{G_\alpha : \alpha \in A\}$ be a family of non-trivial compact groups and $C = \{\alpha \in A : G_\alpha$ is connected$\}$. Then*

$$tgw\left(\prod_{\alpha\in A} G_\alpha\right) = \sqrt[\omega]{|A|}\cdot(|A\setminus C|+1)\cdot\sup\{tgw(G_\alpha) : \alpha\in A\}. \tag{4}$$

Proof. Define $G = \prod_{\alpha\in A} G_\alpha$ and $D = \{\alpha \in A : G_\alpha$ is totally disconnected$\}$. Since $c(G_\alpha)$ are non-trivial if and only of $\alpha \in A\setminus D$, and $c(G) = \prod_{\alpha\in A} c(G_\alpha) \cong \prod_{\alpha\in A\setminus D} c(G_\alpha)$, we have

$$w(c(G)) = w\left(\prod_{\alpha\in A} c(G_\alpha)\right) = (|A\setminus D|+1)\cdot\sup\{w(c(G_\alpha)) : \alpha\in A\}.$$

Combining this with the fact that the function $\sqrt[\omega]{-}$ commutes with suprema, and noting that $\sqrt[\omega]{\tau+1} = \sqrt[\omega]{\tau}$, we get

$$\sqrt[\omega]{w(c(G))} = \sqrt[\omega]{|A\setminus D|}\cdot\sup\{\sqrt[\omega]{w(c(G_\alpha))} : \alpha\in A\}. \tag{5}$$

Note that $|G/c(G_\alpha)| \geq 2$ if and only if $\alpha \in A\backslash C$, so from $G/c(G) = \prod_{\alpha\in A} G_\alpha/c(G_\alpha) \cong \prod_{\alpha\in A\backslash C} G_\alpha/c(G_\alpha)$ we conclude that

$$w(G/c(G)) = w\left(\prod_{\alpha\in A} G_\alpha/c(G_\alpha)\right) = (|A\setminus C|+1)\cdot\sup\{w(G_\alpha/c(G_\alpha)) : \alpha\in A\}. \tag{6}$$

For each $\alpha \in A$, from Theorem 2.3 we get

$$\sqrt[\omega]{w(c(G_\alpha))}\cdot w(G_\alpha/c(G_\alpha)) = tgw(G_\alpha). \tag{7}$$

Since all groups G_α are non-trivial, $D \subseteq A\setminus C$, and so $\sqrt[\omega]{D}\cdot(|A\setminus C|+1) = \omega\cdot(|A\setminus C|+1)$, which implies

$$\sqrt[\omega]{|A\setminus D|}\cdot(|A\setminus C|+1) = \sqrt[\omega]{|A\setminus D|}\cdot\sqrt[\omega]{|D|}\cdot(|A\setminus C|+1) = \sqrt[\omega]{|A|}\cdot(|A\setminus C|+1). \tag{8}$$

Now the conclusion of our theorem follows from Theorem 2.3, (5), (6), (7) and (8). QED

Theorem 7.2 *If the compact group G is a semidirect product $G = N \rtimes G/N$, then $tgw(G) = tgw(N) \cdot tgw(G/N)$.*

Proof. Set for convenience $H = G/N$ and note that $w(G) = w(N) \cdot w(H)$. Let $\pi : G \to H$ be the quotient homomorphism, and let $\pi' : G \to N$ be the continuous projection (note that π' need not be a homomorphism).

We start the proof with the special case when G is connected. Being continuous images of the connected space G, both N and H are connected. Theorem 2.2 yields

$$tgw(G) = \sqrt[\omega]{w(G)} = \sqrt[\omega]{w(N) \cdot w(H)} = \sqrt[\omega]{w(N)} \cdot \sqrt[\omega]{w(H)} = tgw(N) \cdot tgw(H).$$

In the general case, by making use of the continuous projections π and π', we get $c(G) = c(N) \rtimes c(H)$. Hence $tgw(c(G)) = tgw(c(N)) \cdot tgw(c(H)$ by the special case proved above. The isomorphisms

$$G/c(G) \cong (N \rtimes H)/(c(N) \rtimes c(H)) = N/c(N) \rtimes H/c(H)$$

yield $w(G/c(G)) = w(N/c(N)) \cdot w(H/c(H))$. Therefore, from Theorems 2.2 and 2.3 we get

$$tgw(G) = tgw(c(G)) \cdot w(G/c(G)) =$$
$$tgw(c(N)) \cdot tgw(c(H)) \cdot w(N/c(N)) \cdot w(H/c(H)) = tgw(N) \cdot tgw(H).$$

QED

7.2 Monotonicity of the function $tgw(-)$ with respect to closed subgroups

If N is a closed subgroup (or even a subspace) of a topological group G, then $w(N) \leq w(G)$. Our next example demonstrates that the counterpart of this inequality for the topologically generating weight does not hold.

Example 7.3 If τ is an infinite cardinal such that $\tau^\omega > \tau$ and N is a closed totally disconnected subgroup of the compact group $G = \mathbb{T}^{\tau^\omega}$ with $w(N) = \tau^\omega$ (for example, $N = \{x \in \mathbb{T}^{\tau^\omega} : 2x = 0\}$ will do the job), then $tgw(N) = \tau^\omega > \tau \geq \sqrt[\omega]{\tau^\omega} = tgw(G)$ by Theorems 2.1 and 2.2.

In the proof of our next theorem we need the following (perhaps folklore) lemma which is of independent interest. We decided to give a self-contained topological proof of Lemma 7.4 instead of a standard proof using the structure theory of compact connected groups.

Lemma 7.4 *Let G be a compact connected group of uncountable weight. Then for every cardinal κ satisfying $\omega_1 \leq \kappa \leq w(G)$ the following holds:*

(i) G has a connected closed normal subgroup K_κ such that $w(K_\kappa) = \kappa$,

(ii) G has a totally disconnected closed subgroup N_κ such that $w(N_\kappa) = \kappa$.

Proof. We will need the following

Claim 7.4.1 *Assume that $\{H_i : i \in I\}$ is a family of compact metric groups, G is a compact group, $f : \prod_{i\in I} H_i \to G$ is a surjective group homomorphism and κ is a cardinal satisfying $\omega_1 \le \kappa \le |I|$. Moreover, assume that D_i is a dense subset of H_i for every $i \in I$. Then there exist a faithfully indexed family $\{i_\alpha : \alpha \in \kappa\} \subseteq I$ and a family $\{x_\alpha : \alpha \in \kappa\}$ such that $x_\alpha \in D_{i_\alpha}$ and $f(x_\alpha) \notin \overline{\{f(x_\beta) : \beta < \alpha\}}$ whenever $\alpha \in \kappa$.*

Proof. For every $J \subseteq I$ we identify $K_J = \prod_{i\in J} H_i$ with a subgroup of $\prod_{i\in J} H_i$ in a canonical way. By transfinite induction on $\alpha \in \kappa$ we will define $i_\alpha \in I \setminus \{i_\beta : \beta < \alpha\}$ and $x_\alpha \in D_{i_\alpha}$ such that $f(x_\alpha) \notin \overline{\{f(x_\beta) : \beta < \alpha\}}$. Suppose that $\alpha \in \kappa$ and we have already defined the required i_β and x_β for all $\beta < \alpha$. Let $I_\alpha = \{i_\beta : \beta < \alpha\}$ and $J_\alpha = I \setminus I_\alpha$. Then $N_\alpha = f(K_{I_\alpha})$ is a compact (and thus closed) subgroup of G. Moreover, $w(N_\alpha) \le w(K_{I_\alpha}) \le \omega \cdot |\alpha|$ by [9, Theorem 3.1.22]. Since $\omega \cdot |\alpha| < \kappa \le w(G)$, it follows that $w(N_\alpha) < w(G)$. If $f(H_i) \subseteq N_\alpha$ for every $i \in J_\alpha$, then

$$f(K_{J_\alpha}) = f\left(\overline{\bigoplus_{i\in J_\alpha} H_i}\right) \subseteq \overline{N_\alpha} = N_\alpha,$$

and thus $G = f(K_I) = f(K_{I_\alpha} \oplus K_{J_\alpha}) = f(K_{I_\alpha}) + f(K_{J_\alpha}) \subseteq N_\alpha + N_\alpha = N_\alpha$, in contradiction with $w(N_\alpha) < w(G)$. Therefore, $f(H_{i_\alpha}) \setminus N_\alpha \ne \emptyset$ for some $i_\alpha \in J_\alpha$. Since $H_{i_\alpha} \setminus f^{-1}(N_\alpha)$ is a non-empty open subset of H_{i_α}, and D_{i_α} is dense in H_{i_α}, we can pick $x_\alpha \in D_{i_\alpha}$. Now note that

$$f(x_\alpha) \notin N_\alpha = \overline{N_\alpha} \supseteq \overline{\{f(x_\beta) : \beta < \alpha\}}$$

by our construction. QED

By Theorem 3.3, there exists a family $\{H_i : i \in I\}$ of compact connected groups with $|I| = w(G)$ and a continuous surjective group homomorphism $f : \prod_{i\in I} H_i \to G$. For every $i \in I$ let D_i be the set of all elements $x \in H_i$ such that the subgroup $\overline{\langle x\rangle}$ of H_i is totally disconnected. It is well-known that D_i is dense in H_i [20].

Let κ be a cardinal satisfying $\omega_1 \le \kappa \le w(G)$, and let $\{i_\alpha : \alpha \in \kappa\} \subseteq I$ and $\{x_\alpha : \alpha \in \kappa\}$ be as in the conclusion of Claim 7.4.1. Identify $K^* = \prod_{\alpha\in\kappa} H_{i_\alpha}$ and $N^* = \prod_{\alpha\in\kappa} \overline{\langle x_\alpha\rangle}$ with subgroups of $\prod_{i\in I} H_i$ in a canonical way, and note that $N^*_\kappa \subseteq K^*_\kappa$. Being a product of connected compact groups, K^*_κ is a connected compact group, and so $K_\kappa = f(K^*_\kappa)$ is also connected and compact (thus also closed in G). Being a product of totally disconnected compact group, N^*_κ is a totally disconnected compact group, and so $N_\kappa = f(N^*_\kappa)$ is also totally disconnected and compact (thus also closed in G). Moreover, K_κ is a normal subgroup of G.

Since $f(x_\alpha) \notin \overline{\{f(x_\beta) : \beta < \alpha\}}$ whenever $\alpha \in \kappa$, we conclude that $w(\{f(x_\alpha) : \alpha < \kappa\}) \ge \kappa$. From $\{x_\alpha : \alpha < \kappa\} \subseteq N^*_\kappa$ it follows that $\{f(x_\alpha) : \alpha < \kappa\} \subseteq N_\kappa \subseteq K_\kappa$, and so

$$\kappa \le w(\{f(x_\alpha) : \alpha < \kappa\}) \le w(N_\kappa) \le w(K_\kappa) \le w(K^*_\kappa) \le \kappa.$$

QED

In the next theorem we characterize the compact groups G such that the function tgw(-) is monotone with respect to closed subgroups of G.

Theorem 7.5 *Let G be a compact group. Then the following are equivalent:*

(a) $w(c(G)) \leq \omega$,

(b) $tgw(N) = w(N)$ *for all closed subgroups* N *of* G,

(c) $tgw(N) \leq tgw(H)$ *for any pair* N *and* H *of closed subgroups of* G *such that* $N \subseteq H$.

Proof. (a) $\Rightarrow$ (b). Let N be a closed subgroup of G. Since $c(N) \subseteq c(G)$, we have $w(c(N)) \leq w(c(G)) \leq \omega$, and so $tgw(N) = \sqrt[\omega]{w(c(N))} \cdot w(N/c(N)) = \omega \cdot w(N/c(N)) = w(N/c(N)) = w(N)$ by Theorem 2.3.

(b) $\Rightarrow$ (c). If N and H are as in the assumption of (c), then (b) implies $tgw(N) = w(N) \leq w(G) = tgw(G)$.

(c) $\Rightarrow$ (a) Assume that $w(c(G)) > \omega$. According to Lemma 7.4, there exist closed connected subgroup H of $c(G)$ and a closed totally disconnected subgroup N of $c(G)$ such that $N \subseteq H$ and $w(N) = w(H) = \omega_1$. Now $tgw(H) = \omega < \omega_1 = tgw(N)$ by Theorems 2.2 and 2.1. This contradicts (c). QED

Item (i) of our next corollary strengthens Theorem 2.1.

Corollary 7.6 *Let G be a compact group with $w(c(G)) \leq \omega$. Then:*
(i) $tgw(G) = w(G)$, *and*
(ii) $tgw(N) \leq tgw(G)$ *for every closed subgroup* N *of* G.

Another easy case of monotonicity of $tgw(-)$ is provided by the following

Theorem 7.7 *If N is a closed subgroup of a compact Abelian group G such that $c(N) = c(G)$, then $tgw(N) \leq tgw(G)$.*

Proof. Theorem 2.3 yields

$$tgw(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) = \sqrt[\omega]{w(c(N))} \cdot w(G/c(G)) \geq$$

$$\sqrt[\omega]{w(c(N))} \cdot w(N/c(N)) = tgw(N),$$

where the inequality $w(G/c(G)) \geq w(N/c(N))$ follows from the fact that $N/c(N) = N/c(G)$ is isomorphic to a subgroup of $G/c(G)$. QED

Yet another instance of monotonicity of $tgw(-)$ is contained in Theorem 7.2 which shows that $tgw(N) \leq tgw(G)$ whenever a closed subgroup N of a compact group G is a semidirect summand of G.

Our next example demonstrates that Theorem 7.5 fails when one replaces "closed subgroups" in it with "closed *normal* subgroups".

Example 7.8 Let L be a simple connected compact Lie group with trivial center. Then $tgw(N) \leq tgw(H)$ for every cardinal τ and for every pair $N \leq H$ of closed normal subgroups of L^τ. Indeed, every closed normal subgroup of L^τ has a "rectangular" form L^λ, so monotonicity of the root function and Theorem 2.2 apply.

Remark 7.9 For every infinite cardinal τ, the group L^τ from Example 7.8 has the following property: Every totally disconnected closed *normal* subgroup of L^τ is trivial. This shows that in item (ii) of Lemma 7.4 the subgroup N_κ need not be normal.

In conclusion we show that, contrary to the case of compact groups with small (metrizable) connected component considered above, in the case of the connected compact groups G the function tgw behaves in an "anti-monotone" way.

Lemma 7.10 *Let G be a compact connected group. Then $tgw(N) \geq tgw(G)$ for every closed subgroup N of G with $w(N) = w(G)$.*

Proof. If $w(c(N)) = w(N)$, then $tgw(N) \geq \sqrt[\omega]{w(c(N))} = \sqrt[\omega]{w(c(G))} = tgw(G)$. If $w(c(N)) < w(N)$, then $w(N/c(N)) = w(N) = w(G)$, and therefore $tgw(N) \geq w(N/c(N)) = w(G) \geq tgw(G)$. QED

7.3 The "three space problem"

Theorem 7.11 *For every closed normal subgroup N of a compact group G the inequality*

$$tgw(G) \leq tgw(N) \cdot tgw(G/N) \tag{9}$$

holds.

Proof. Define $G_1 = G/c(G)$ and note that $tgw(G) \geq w(G_1)$ by Theorem 2.1. We consider two cases.

Case 1. $tgw(G) = w(G_1)$. Let $f : G \to G_1$ be the canonical homomorphism and $N_1 = f(N)$. Then $tgw(N) \geq tgw(N_1)$ by Fact 4.9. Since G_1 is totally disconnected, also its subgroup N_1 is totally disconnected, so $tgw(N_1) = w(N_1)$ by Theorem 2.1. Therefore, $tgw(N) \geq w(N_1)$.

Since G_1 is a compact totally disconnected group, so is its quotient group G_1/N_1, and Theorem 2.1 once again gives $tgw(G_1/N_1) = w(G_1/N_1)$. From $G_1/N_1 \cong G/(c(G) \cdot N)$ it follows that G_1/N_1 is isomorphic to a quotient group of G/N, and Fact 4.9 implies

$$tgw(G/N) \geq tgw(G_1/N_1) = w(G_1/N_1). \tag{10}$$

Combining the inequality $tgw(N) \geq w(N_1)$ obtained above with (10) we get

$$tgw(N) \cdot tgw(G/N) \geq w(N_1) \cdot w(G_1/N_1) = w(G_1) = tgw(G),$$

where the last equality is due to our hypothesis.

Case 2. $tgw(G) > w(G_1)$. Define $\alpha = w(c(G))$. If $\alpha \leq w(G_1)^\omega$, then $\sqrt[\omega]{\alpha} \leq w(G_1)$, and Theorem 2.3 gives $tgw(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) = \sqrt[\omega]{\alpha} \cdot w(G_1) \leq w(G_1)$, a contradiction. Therefore

$$\alpha > w(G_1)^\omega \text{ and } tgw(G) = \sqrt[\omega]{\alpha} = tgw(c(G)), \tag{11}$$

according to Theorem 2.3. In particular, $\alpha > w(G_1)$, hence $w(G) = w(c(G)) \cdot w(G_1) = \alpha \cdot w(G_1) = \alpha$. Now we consider two sub-cases.

Case 2(a). $w(N \cap c(G)) = \alpha$. Then $w(N) = \alpha$, and so $tgw(c(G)) \leq tgw(N \cap c(G))$ by Lemma 7.10. Hence, $tgw(G) \leq tgw(N \cap c(G))$ by (11). Since $c(N) = c(N \cap c(G))$, Theorem 7.7 implies $tgw(N \cap c(G)) = tgw(N)$. Consequently, $tgw(G) \leq tgw(N)$, which yields (9).

Case 2(b). $w(N \cap c(G)) < \alpha$. Then

$$w(c(G)/(N \cap c(G)) = w(c(G)) = \alpha. \tag{12}$$

By the second isomorphism theorem for topological groups [10, (2.2)], we have $G/(c(G) \cdot N) \cong (G/N)/((c(G) \cdot N)/N)$, which yields that $(G/N)/(c(G) \cdot N)/N$ is totally disconnected, and so the connected subgroup $(c(G) \cdot N)/N$ of the group G/N coincides with its connected component $c(G/N)$. Therefore,

$$c(G/N) \cong (c(G) \cdot N)/N \tag{13}$$

and

$$(G/N)/c(G/N) \cong (G/N)/(c(G) \cdot N)/N)) \cong G/(c(G) \cdot N). \tag{14}$$

Since $c(G)/(N \cap c(G)) \cong (c(G) \cdot N)/N$, from (12) and (13) we conclude that $w(c(G/N)) = \alpha$. Since $G/(c(G) \cdot N)$ is isomorphic to a quotient of $G_1 = G/c(G)$, from (14) we have $w((G/N)/c(G/N)) \leq w(G_1)$. Combining this with (11), we get $w((G/N)/c(G/N))^\omega \leq w(G_1)^\omega < \alpha = w(c(G/N))$, and hence $\sqrt[\omega]{w(c(G/N))} > w((G/N)/c(G/N))$. ¿¿From Theorem 2.3 and (11) we finally get

$$tgw(G/N) = \sqrt[\omega]{w(c(G/N))} \cdot w((G/N)/c(G/N)) = \sqrt[\omega]{w(c(G/N))} = \sqrt[\omega]{\alpha} = tgw(G),$$

which yields (9). QED

Remark 7.12 Example 7.3 shows that the inequality in Theorem 7.11 cannot be strengthened to the equality $tgw(G) = tgw(N) \cdot tgw(G/N)$. This is also in sharp contrast with the case of the weight because for a closed normal subgroup N of a compact group G the equality $w(G) = w(N) \cdot w(G/N)$ always holds.

8 Proof of the equality $tgw(G) = seq(G)$

Obviously, the inequality $tgw(G) \leq seq(G)$ holds for every group G for which $seq(G)$ is defined. In our next theorem we prove that, rather unexpectedly, this inequality becomes an equality for compact groups G:

Theorem 8.1 *$tgw(G) = seq(G)$ for every compact group G.*

Proof. This theorem is a trivial consequence of a deep result by Hofmann and Morris [13, Proposition 12.29] and our principal Theorem 2.3. Indeed, let G be a compact group. Then $seq(G) \leq \sqrt[\omega]{w(c(G))} \cdot w(G/c(G))$ by [13, Proposition 12.29], which combined with our Theorem 2.3 yields $seq(G) \leq tgw(G)$. The reverse inequality $tgw(G) \leq seq(G)$ holds trivially.

For the sake of completeness of exposition, as well as for the benefit of the curious reader, we are going to show now how a slight modification of our existing proofs yields a self-contained direct proof of this theorem that makes no recourse to the powerful (but highly sophisticated) techniques from the monograph [13].

Claim 8.1.1 [12, Lemma 4.3] *If $\pi : G \to H$ is a continuous homomorphism of a topological group G onto a topological group H, then $seq(H) \leq seq(G)$.*

Proof. Let X be a super-sequence in G topologically generating G such that $|X| = seq(G)$. Then $\pi(X)$ is a super-sequence in H (Fact 4.3) topologically generating H. Therefore, $seq(H) \leq |\pi(X)| \leq |X| \leq seq(G)$. QED

Claim 8.1.2 [13, Theorem 12.25] *If G is a compact connected group, then $seq(G) \leq \sqrt[\omega]{w(G)}$.*[3]

Proof. Let $\tau = \sqrt[\omega]{w(G)}$. Replacing the reference to Lemma 4.9 by the reference to Claim 8.1.1 in the proof of Lemma 6.5, one gets $seq(G) \leq seq(H) \leq seq(K^{\tau^\omega}) \leq \tau$. QED

Since $seq(G) \leq w(G)$, the following claim is a particular case of [13, Proposition 12.20]:

Claim 8.1.3 *If H is a closed normal subgroup of a compact group G, then $seq(G) \leq seq(H) \cdot w(G/H)$.*

Proof. A trivial modification of the proof of Lemma 4.10 will do. Since $seq(H)$ is defined, we can replace the original set K from the proof of Lemma 4.10 by a super-sequence in H topologically generating H such that $|K| = seq(H)$. Fact 4.4 allows us to assume, without loss of generality, that K converges to e provided that K is infinite. Now one can easily check, using Fact 4.2, that $X' \cup K$ is a super-sequence in G. Since $|X' \cup K| \leq seq(H) \cdot w(G/H)$ and $X' \cup K$ topologically generates G, we are done. QED

From Claims 8.1.3, 8.1.2 and Theorem 2.3, one gets

$$seq(G) \leq seq(c(G)) \cdot w(G/c(G)) \leq \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) = tgw(G).$$

The reverse inequality $tgw(G) \leq seq(G)$ holds trivially. QED

Remark 8.2 Mel'nikov [17] essentially proved that $seq(G) = w(G)$ for a totally disconnected compact group G. Theorem 8.1 implies that this result of Mel'nikov is "equivalent" to our Theorem 2.1.

[3] Actually, the equality $seq(G) = \sqrt[\omega]{w(G)}$ has been proved in [13, Theorem 12.25].

9 Applications to Hofmann-Morris' generating rank $s(G)$

9.1 General case

One easily sees that a suitable subset of a compact group G must be a super-sequence. Therefore, $seq(G) = \omega \cdot s(G)$ for a compact group G, i.e. $s(G)$ and $seq(G)$ coincide whenever $s(G)$ is infinite. Combining this observation with Theorem 8.1 yields

Corollary 9.1 *$\omega \cdot s(G) = tgw(G)$ for a compact group G.*

This corollary makes all our results about $tgw(G)$ immediately applicable to the study of the generating rank $s(G)$ of a compact group G. Let us explicitly mention only one general result and several particular consequences:

Corollary 9.2 *If G is a compact group of infinite generating rank $s(G)$, then $s(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G))$, where $c(G)$ is the connected component of G.*

Proof. Since $s(G) = seq(G)$ when $s(G)$ is infinite, it remains only to combine Theorems 2.3 and 8.1. QED

The following property of $s(G)$ is one of the main results of [12] (cf. Theorem 4.14):

Corollary 9.3 *$s(G)^\omega = w(G)^\omega$ for every compact group G with $s(G) \geq 2$.*

Proof. If $s(G) \geq 2$ is finite, then $seq(G) = \omega \cdot s(G) = \omega$ and thus $s(G)^\omega = 2^\omega$. Since G has a finitely generated subgroup, $w(G) \leq 2^\omega$, and so $s(G)^\omega = 2^\omega = w(G)^\omega$. If $s(G)$ is infinite, then $s(G) = seq(G)$, and so $s(G) = tgw(G)$ by Theorem 8.1. Now $s(G)^\omega = tgw(G)^\omega = w(G)^\omega$ by Corollary 2.4. QED

Recall that (†) denotes a rather mild additional set-theoretic assumption defined in the paragraph preceding Theorem 2.8. We refer the reader to the text following Theorem 2.10 for the discussion of relative strength of (†) and its consistency with ZFC. Under (†), we obtain a simple rule for computing the generating rank $s(G)$ of compact groups G with $w(G) > \mathfrak{c}$.

Corollary 9.4 *Assume (†), and let G be a compact group such that $w(G) > \mathfrak{c}$. If $w(G/c(G)) < w(G) = \rho^+$, where ρ is a limit cardinal with $\mathrm{cf}(\rho) = \omega$, then $s(G) = \rho$, and thus $s(G)^+ = w(G)$. Otherwise, $s(G) = w(G)$.*

Proof. The assumption $w(G) > \mathfrak{c}$ yields $s(G) > \omega$. Therefore, $s(G) = tgw(G)$ by Corollary 9.1. Now the conclusion follows from Theorem 2.8. QED

Hofmann and Morris raised in [12, p. 190] the problem [4] of computation of the precise value of the generating rank $s(N \rtimes H)$ of the semidirect product $G = N \rtimes H$ in terms of the generating ranks $s(N)$ and $s(H)$. Our next corollary completely resolves this problem in case of infinite generating ranks. We note that [12, Lemma 4.6] gives a particular property that, even if put in its strongest form, could only prove the inequality $seq(N \rtimes H)w(H) \geq seq(N)$.

Corollary 9.5 *Let G be compact group and let N be a closed normal subgroup of G such that the generating ranks $s(N)$ and $s(G/N)$ are infinite. If the compact group G is a semidirect product $G = N \rtimes G/N$, then $s(G) = s(N) \cdot s(G/N)$.*

Proof. Since $s(G) \geq s(G/N)$ is also infinite, we have $s(G) = seq(G)$, $s(N) = seq(N)$ and $s(N \rtimes G/N) = seq(N \rtimes G/N)$. Now the conclusion of our theorem follows from Theorems 7.2 and 8.1. QED

We comment further on the relation of our results to those in [13, Chapter 12].

The equality $seq(G) = \sqrt[\omega]{w(G)}$ for a compact connected group G was proved for the first time in [13, Theorem 12.25] with the use of free compact topological groups. It follows from our Theorems 2.2 and 8.1.

The inequality $seq(G) \leq \sqrt[\omega]{w(c(G))} \cdot w(G/c(G))$ for a compact group G was proved in [13, Proposition 12.29] by means of free compact topological groups and the powerful [13, Global Cross Section Theorem for Totally disconnected base spaces 10.35]. In fact, the stronger equality $seq(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G))$ immediately follows from our Theorems 8.1 and 2.3.

The "three space inequality" $seq(G) \leq seq(N) \cdot seq(G/N)$ for a closed normal subgroup of a compact group G has been proved in [13, Proposition 12.20] by means of [13, Global Cross Section Theorem for Totally disconnected base spaces 10.35]. It follows from our Theorems 7.11 and 8.1.

Remark 9.6 (a) Proposition 12.30 (iv) from [13] establishes $\omega \cdot s(G) = w(G/c(G))$ for compact groups G with $w(G) \leq \mathfrak{c}$ and infinite quotient $G/c(G))$. This follows also from Theorems 2.3 and 8.1, since $\sqrt[\omega]{w(G)} = \sqrt[\omega]{w(c(G))} = \omega$ for such a group G.

(b) Assume that $s(G)$ is finite. If G is connected, then $s(G) = 1$ if and only if G is Abelian [13, Proposition 12.30(ii)]; otherwise, $s(G) = 2$ [13, Proposition 12.30(iii)]. Note that $s(G/c(G)) \leq s(G)$, so $s(G/c(G))$ is also finite. Now we know that $s(G) \leq s(c(G)) + s(G/c(G))$ [13, Proposition 12.20], where $s(c(G))$ is either 1 or 2. This establishes the inequality $s(G/c(G)) \leq s(G) \leq s(G/c(G)) + 2$ proved in [13, Proposition 12.30(v)] under the restriction $1 < |G/c(G)| < \omega$.

Remark 9.7 Let G be a compact group of finite generating rank $s(G)$. Then $w(G/c(G)) \leq \omega$ by Corollary 9.1 and Theorem 2.3. Moreover, $w(G) = w(c(G)) \cdot w(G/c(G)) = w(c(G)) \cdot \omega = w(c(G))$. Clearly, $w(G) \leq \mathfrak{c}$ because G is separable.

[4] Indeed, one reads in p. 190: "Not too much can be said (about the sequential rank of semidirect products of compact groups)".

9.2 Abelian case

Let $\mathbb{P}$ denote the set of prime numbers. For $p \in \mathbb{P}$ we denote by $\mathbb{Z}_p$ the compact group of p-adic integers. Recall also that $\widehat{G}$ denotes the Pontryagin dual of G.

Our next result characterizes the compact Abelian groups G of finite generating rank $s(G)$.

Proposition 9.8 *Let G be a compact Abelian group and $n \in \mathbb{N}$. Then the following are equivalent:*

(a) $s(G) \leq n$,

(b) $w(G) \leq \mathfrak{c}$ and $r_p(G/pG) \leq n$ for every prime p,

(c) $r(\widehat{G}) \leq \mathfrak{c}$ and $r_p(\widehat{G}) \leq n$ for every prime p,

(d) G is isomorphic to a quotient group of the group $M_n = \widehat{\mathbb{Q}}^{\mathfrak{c}} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^n$.

Proof. (a) $\Rightarrow$ (b). Obviously, (a) implies $w(G) \leq \mathfrak{c}$ and $s(G/pG) \leq n$ for every prime p. Since the group G/pG has exponent p, the latter inequality is equivalent to $r_p(\widehat{G}) \leq n$. (b) $\Rightarrow$ (c). By standard properties of Pontryagin duality (see [10, 13]), one has $r_p(\widehat{G}) = r_p(G/pG)$ and $w(G) = \omega \cdot r(\widehat{G})$. (Actually, this proves the equivalence (b) $\Leftrightarrow$ (c).)

(c) $\Rightarrow$ (d). Let D_n denote the discrete group $\mathbb{Q}^{(\mathfrak{c})} \oplus \left(\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^n\right)$, where $\mathbb{Q}^{(\mathfrak{c})}$ is the direct sum of $\mathfrak{c}$ many copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ is the Prüfer p-group. A standard exercise in algebra using our hypothesis (c) allows one to find an injective homomorphism (=monomorphism) $f : \widehat{G} \to D_n$. (See, for example, [4, Section 3] for exposition of relevant ideas.) Taking the Pontryagin duals, we conclude that G is isomorphic to a quotient group of the group $\widehat{D_n} \cong M_n$.

(d) $\Rightarrow$ (a). Note that $\mathbb{T}^n$ equipped with the discrete topology is isomorphic to D_n. Let $\iota : D_n \to \mathbb{T}^n$ be an isomorphism. Taking the Pontryagin duals, we conclude that there exists of a homomorphism $\widehat{\iota} : \mathbb{Z}^n \cong \widehat{\mathbb{T}^n} \to \widehat{D_n} \cong M_n$ with dense image ([3, Lemma 3.2.4]). This proves that $s(G) \leq s(M_n) \leq s(\mathbb{Z}^n) = n$. QED

As a final application, we can derive the formula from [13, Proposition 12.17] computing $s(G)$ in terms of the dual group $\widehat{G}$:

Corollary 9.9 *If G is a compact Abelian group, then*

$$s(G) = \min\{\tau : (\tau+1)^\omega \geq r(\widehat{G}) \text{ and } r_p(\widehat{G}) \leq \tau \text{ for every prime } p\}. \tag{15}$$

Proof. If $s(G)$ is finite, then (15) follows from Proposition 9.8. Assume now that $s(G)$ is infinite. By the standard properties of Pontryagin duality (see [10, 13]), one has $\widehat{c(G)} = \widehat{G}/t(\widehat{G})$ and $w(c(G)) = |\widehat{c(G)}| = |\widehat{G}/t(\widehat{G})| = \omega \cdot r(\widehat{G})$, as well as

$\widehat{G/c(G)} = t(\widehat{G})$ and $w(G/c(G)) = \omega \cdot |\widehat{G/c(G)}| = \omega \cdot |t(\widehat{G})|) = \omega \cdot \sup_{p\in\mathbb{P}}\{r_p(\widehat{G})\}$. Together with Corollary 9.2 this yields

$$s(G) = \sqrt[\omega]{w(c(G))} \cdot w(G/c(G)) = \sqrt[\omega]{r(\widehat{G})} \cdot \sup_{p\in\mathbb{P}}\{r_p(\widehat{G})\} =$$

$$\omega \cdot \min\{\tau : (\tau+1)^\omega \geq r(\widehat{G}) \text{ and } r_p(\widehat{G}) \leq \tau \text{ for every prime } p\}. \tag{16}$$

To see that this coincides with (15), it suffices to remove ω from (16). This is possible since our hypothesis $s(G) \geq \omega$ and Proposition 9.8 yield that either $w(c(G)) = r(\widehat{G}) > \mathfrak{c}$ or $\sup_{p\in\mathbb{P}}\{r_p(\widehat{G})\} \geq \omega$. QED

Remark 9.10 The reader may wonder why in our Definition 4.7 we (apparently artificially) insist on having $seq(G)$ always infinite. There is a good reason for doing so besides the trivial convenience that allows for much nicer formulation of the results (compare, for example, Theorem 8.1 with Corollary 9.1, or Theorem 2.3 with Corollary 9.2).

(a) It follows from our Theorems 8.1 and 2.2 that $tgw(G) = seq(G)$ is a function of $w(G)$ when G is a compact connected group. Quite the contrary, the generating rank $s(G)$ is *not* a function of $w(G)$ for a compact connected group G. To see this, note that, according to Remark 9.6(b), two connected compact groups G and H of the *same weight* $\tau \leq \mathfrak{c}$ may have $s(G) = 1$ and $s(H) = 2$ if G is Abelian and H is non-Abelian.

(b) Theorem 2.1 says that $tgw(G)$ is also a function of $w(G)$ when G is a compact totally disconnected group. On the other hand, for every natural number n, there exists a totally disconnected compact group H_n such that $w(H_n) = \omega$ (and thus $tgw(H_n) = \omega$ by Theorem 2.1) and $s(H_n) = n$. (Indeed, Proposition 9.8 yields that $H_n = \mathbb{Z}_2^n$ is such a group.) Therefore, $s(G)$ is *not* a function of $w(G)$ for a compact totally disconnected group G.

(c) In the general case, it follows from our Theorems 8.1 and 2.3 that $tgw(G) = seq(G)$ is a function of two variables $w(c(G))$ and $w(G/c(G))$. Let us see that the generating rank $s(G)$ is *not* a function of $w(c(G))$ and $w(G/c(G))$. Indeed, for every pair (τ, n), where $\tau \leq \mathfrak{c}$ is an infinite cardinal and n is a natural number, there exists a compact Abelian group $G_{\tau,n}$ with $w(c(G_{\tau,n})) = w(G_{\tau,n}) = \tau$, $w(G_{\tau,n}/c(G_{\tau,n})) = \omega$ and $s(G_{\tau,n}) = n$. Indeed, $G_{\tau,n} = \mathbb{T}^\tau \times \mathbb{Z}_2^n$ is such a group by Proposition 9.8.

Remark 9.11 All results from Section 2, as well as Theorem 8.1, were exposed by the first named author in March 1998 at the Symposium on General and Geometric Topology in Kyoto, and by the second named author in August 2000 at the First Turkish International Topology Conference in Istanbul. Furthermore, these results were included (without any proofs) in the survey [2].

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