FORCING HEREDITARILY SEPARABLE COMPACT-LIKE GROUP TOPOLOGIES ON ABELIAN GROUPS

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ABSTRACT. Let $\mathfrak{c}$ denote the cardinality of the continuum. Using forcing we produce a model of $ZFC + CH$ with $2^{\mathfrak{c}}$ "arbitrarily large" and, in this model, obtain a characterization of Abelian groups G (necessarily of size at most $2^{\mathfrak{c}}$) which admit:

(i) a hereditarily separable group topology,

(ii) a group topology making G into an S-space,

(iii) a hereditarily separable group topology that is either precompact, or pseudocompact, or countably compact (and which can be made to contain no non-trivial convergent sequences),

(iv) a group topology making G into an S-space that is either precompact, or pseudocompact, or countably compact (and which also can be made without non-trivial convergent sequences, if necessary).

As a by-product, we completely describe the algebraic structure of Abelian groups of size at most $2^{\mathfrak{c}}$ which possess, at least consistently, a countably compact group topology (without non-trivial convergent sequences, if desired).

We also put to rest a 1980 problem of van Douwen about the cofinality of the size of countably compact Abelian groups.

Dedicated to the memory of Zoltán T. Balogh

All group topologies in this paper are considered to be Hausdorff (and thus Tychonoff). Recall that a topological space X is:

Lindelöf if every open cover of X has a countable subcover,

(countably) compact if every (countable) open cover of X has a finite subcover,

pseudocompact if every real-valued continuous function defined on X is bounded, and

separable if X has a countable dense subset.

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It is well-known that compact $\to$ countably compact $\to$ pseudocompact, and "pseudocompact + Lindelöf" $\leftrightarrow$ compact.

Recall that a topological group G is *precompact*, or *totally bounded*, if G is (topologically and algebraically isomorphic to) a subgroup of some compact group. Pseudocompact groups are precompact [8], so we have a somewhat longer chain

$$\text{compact} \to \text{countably compact} \to \text{pseudocompact} \to \text{precompact}$$

of compactness-like conditions for topological groups.

A space X is called *hereditarily separable* if every subspace of X is separable (in the subspace topology), and X is said to be *hereditarily Lindelöf* if every subspace of X is Lindelöf (in the subspace topology). An *S-space* is a hereditarily separable regular space that is not Lindelöf [45]. We refer the reader to [46, 47, 32, 55] for known results about the problem of the existence of S-spaces.

1. Motivation

Our results originate in three diverse areas of mathematics.

The first source of inspiration comes from the celebrated theory of S-spaces in set-theoretic topology, and especially, a famous 1975 example of Fedorčuk of a hereditarily separable compact space of size $2^{\mathfrak{c}}$. In our paper we completely characterize Abelian groups that admit a group topology making them into an S-space, and we produce the "best possible analogues" of the Fedorčuk space in the category of topological groups. As it turns out, a vast majority of Abelian groups admit group topologies with properties similar to that of the Fedorčuk example.

The second origin lies in topological algebra, where we were motivated by the problem of which Abelian groups admit a countably compact group topology. We completely describe, albeit consistently, the algebraic structure of Abelian groups of size at most $2^{\mathfrak{c}}$ that admit a countably compact group topology.

Our third motivation comes from the theory of cardinal invariants in general topology. We resolve completely a 1980 problem of van Douwen about the cofinality of $|G|$ for a countably compact group G in the case of Abelian groups.

We will now address all three sources of our motivation in detail.

1.1. S-groups à la Fedorčuk. Recall that $|Y| \leq \mathfrak{c}$ for a hereditarily Lindelöf Hausdorff space Y [1], and $|X| \leq 2^{\mathfrak{c}}$ for a separable Hausdorff space X [43]. It is natural to ask whether the last inequality can be strengthened to $|X| \leq \mathfrak{c}$ for a *hereditarily* separable regular space X. If there are no S-spaces, then every hereditarily separable regular space X is hereditarily Lindelöf, and therefore $|X| \leq \mathfrak{c}$ by the result cited above. Todorčević has proved the consistency with ZFC that S-spaces do *not* exist ([54], see also [55]). Therefore, in Todorčević's model of ZFC, hereditarily separable regular spaces have size at most $\mathfrak{c}$. A first consistent example of a hereditarily separable Tychonoff space of size $2^{\mathfrak{c}}$ has been found by Hajnal and Juhász [26]. (An exposition of their forcing construction can also be found in [39].)

Two years later Fedorčuk [22] produced the strongest known example up to date using his celebrated inverse spectra with fully closed maps (see also [23]):

Example 1.1. The existence of the following "Fedorčuk space" X is consistent with ZFC plus CH:
(i) $|X| = 2^{\mathfrak{c}}$,
(ii) X is hereditarily separable,
(iii) X is compact, and
(iv) if F is an infinite closed subset of X, then $|F| = |X|$; in particular, X does not contain non-trivial convergent sequences.

The main goal of this paper is to address the question of the existence of "Fedorčuk space" in the context of topological groups. That is, given a group G, we wonder if it is possible to find a hereditarily separable Hausdorff group topology on G having properties that "Fedorčuk space" has. Since we want to get a hereditarily separable topology on G, we have to restrict ourselves to groups G of size at most $2^{\mathfrak{c}}$. One naturally expects that the presence of algebra may produce additional restrictions on how good a Fedorčuk type group can be. And this is indeed the case.

First of all, one is forced to relax somewhat the compactness condition from item (iii) of Example 1.1 because of two fundamental facts about compact groups:

Fact 1.2. *(i) Infinite compact groups contain non-trivial convergent sequences.*
(ii) Compact hereditarily separable groups are metrizable.

Both facts are folklore and follow from the following result of Hagler, Gerlits and Efimov: An infinite compact group G contains a copy of the Cantor cube $\{0,1\}^{w(G)}$, where $w(G)$ is the weight of G. An elementary proof of this theorem, together with some historical discussion, can be found in [49].

Recall that a space X is *initially ω_1-compact* if every open cover of size $\leq \omega_1$ has a finite subcover. Item (i) of Fact 1.2 is no longer valid, at least consistently, if one replaces "compact" by "initially ω_1-compact" in it: It is consistent with ZFC that there exists an initially ω_1-compact Hausdorff group topology without non-trivial convergent sequences on the free Abelian group of size $\mathfrak{c}$. This result is announced, with a hint at a proof, in [56].

However, item (ii) of Fact 1.2 remains valid if one replaces "compact" by "initially ω_1-compact" in it, see [2]. This means that countable compactness appears to be the *strongest* compactness type property among weakenings of classical compactness for which one may hope to obtain hereditarily separable group topologies, and indeed, consistent examples of hereditarily separable countably compact groups (without non-trivial convergent sequences) are known in the literature [27, 52, 38]. This perfectly justifies countable compactness as our strongest compactness condition of choice when working with hereditarily separable groups.

Second, we will have to restrict ourselves to Abelian groups because in the non-commutative case there are groups (of small size) that do not admit *any* countably compact or separable group topology, as follows from our next result:

Proposition 1.3. *Let X be a set and $S(X)$ the symmetric group of X.*[1] *Then:*
(i) $S(X)$ does not admit a separable group topology unless X is countable,
(ii) $S(X)$ admits no countably compact group topology when X is infinite, and
(iii) $S(X)$ does not admit a Lindelöf group topology unless X is countable.

Proof. We equip $S(X)$ with the topology of pointwise convergence on X, i.e. the topology $\mathcal{T}_p$ generated by the family $\{U(f,F) : f \in S(X), F \in [X]^{<\omega}\}$ as a base, where $U(f,F) = \{g \in S(X) : g(x) = f(x) \text{ for all } x \in F\}$. It is easy to see that $\mathcal{T}_p$ is a group topology.

Assume that X is an infinite set. For a fixed $x \in X$, the stabilizer $S_x = \{\sigma \in S(X) : \sigma(x) = x\} = U(id_X, \{x\})$ of x is a $\mathcal{T}_p$-open subgroup of $S(X)$ of index $|X|$, and hence it produces an open cover of $S(X)$ by pairwise disjoint sets (obtained by taking appropriate unions of cosets of S_x) without a subcover of size (strictly) less than $|X|$. It follows that the space $(S(X), \mathcal{T}_p)$ is not countably compact, and also is neither separable nor Lindelöf when $|X| > \omega$.

It is known that $\mathcal{T}_p$ is a minimal element in the lattice of all (Hausdorff) group topologies on $S(X)$, i.e. $\mathcal{T}_p \subseteq \mathcal{T}$ for every (Hausdorff) group topology $\mathcal{T}$ on $S(X)$ [25]. This easily yields the conclusion of all three items of our proposition. □

It follows from the above proposition that, for an uncountable set X, the symmetric group $S(X)$ admits neither a separable, nor a countably compact, nor a Lindelöf group topology.[2] Furthermore, free groups never admit countably compact group topologies ([12, Theorem 4.7]; see also [14, Corollary 5.14]).

Third, algebraic restrictions prevent us from getting the full strength of item (iv), as our next example demonstrates:

Example 1.4. Let $G = \mathbb{Z}(2)^{(\mathfrak{c})} \oplus \mathbb{Z}^{(2^{\mathfrak{c}})}$ be the direct sum of the Boolean group $\mathbb{Z}(2)^{(\mathfrak{c})}$ of size $\mathfrak{c}$ and the free Abelian group $\mathbb{Z}^{(2^{\mathfrak{c}})}$ of size $2^{\mathfrak{c}}$. We claim that, *for any Hausdorff group topology on G, there exists a closed (in this topology) infinite set F such that $|F| < |G|$.* In fact, $F = \mathbb{Z}(2)^{(\mathfrak{c})} \subseteq G$ is such a set. Indeed, $|F| = \mathfrak{c} < 2^{\mathfrak{c}} = |G|$, so it remains only to note that F is an *unconditionally closed* subset of G in Markov's sense [42]; that is, F is closed in every Hausdorff group topology on G. The latter

[1] That is, $S(X)$ is a set of bijections of X onto itself with the composition of maps as multiplication.

[2] In particular, no group $S(X)$ admits a Hausdorff *group* topology that makes it into an S-space. This should be compared with substantial difficulties one has to overcome to produce a model of ZFC in which there are no *S-spaces*. Furthermore, no group $S(X)$ admits a Hausdorff *group* topology that makes it into an L-space (i.e., a hereditarily Lindelöf but not (hereditarily) separable space). This should be compared with the fact that the consistency of the non-existence of *L-spaces* is a well-known problem of set-theoretic topology that remains unresolved.

follows from the fact that $F = \{x \in G : 2x = 0\}$ is the preimage of the (closed!) set $\{0\}$ under the continuous map that sends x to $2x$.

We note that our Theorem 2.7 implies that, in an appropriate model of ZFC, the group G from the example above does admit a hereditarily separable countably compact group topology without non-trivial convergent sequences. So the best we can hope for in our quest for Fedorčuk type group G is to require that G satisfies the second, weaker, condition from item (iv) of Example 1.1, i.e. that G *does not have any non-trivial convergent sequences*.

1.2. Algebraic structure of countably compact Abelian groups. Halmos [28] showed that the additive group of real numbers can be equipped with a compact group topology and asked which Abelian groups admit compact group topologies. Halmos' problem seeking a complete description of the algebraic structure of compact Abelian groups contributed substantially to the development of the Abelian group theory, particularly through the introduction of the algebraically compact groups by Kaplansky [35]. This problem has been completely solved in [29, 31].

The counterpart of Halmos' problem for pseudocompact groups asking which Abelian groups can be equipped with a pseudocompact group topology was attacked in [4, 12, 13, 5, 6, 14] and the significant progress has been summarized in the monograph [14]. Recall also that every Abelian group admits a precompact group topology [7].

The question of which Abelian groups admit a countably compact group topology appears to be much more complicated. After a series of scattered results [27, 19, 52, 38, 17, 57] a complete description of the algebraic structure of countably compact Abelian groups of size at most $\mathfrak{c}$ under Martin's Axiom MA has been recently obtained in [18]: MA implies that an Abelian group G of size at most $\mathfrak{c}$ admits a countably compact group topology if and only if it satisfies both **PS** and **CC**, two conditions introduced in Definition 2.3 below. (In particular, every torsion-free Abelian group of size $\mathfrak{c}$ admits a countably compact group topology under MA [53].) In our Theorem 2.7 and Corollary 2.17(ii) we substantially extend this result by proving that, at least consistently, the conjunction of **PS** and **CC** is both a necessary and a sufficient condition for the existence of a countably compact group topology on an Abelian group G of size at most $2^{\mathfrak{c}}$. Moreover, we get both hereditary separability and absence of non-trivial convergent sequences for our group topology as a bonus.

This "jump" from $\mathfrak{c}$ to $2^{\mathfrak{c}}$ is an essential step forward. Indeed, amazingly little is presently known about the existence of countably compact group topologies on groups of cardinality greater than $\mathfrak{c}$. Using a standard closing-off argument van Douwen [20] showed that every infinite Boolean group of size $\kappa = \kappa^{\omega}$ admits a countably compact group topology and his argument can easily be extended to Abelian groups of prime exponent. It is consistent with ZFC that the Boolean

group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa \leq 2^{\mathfrak{c}}$ [58]. (Here $2^{\mathfrak{c}}$ can be made "arbitrary large".) It is also consistent with ZFC that the free Abelian group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa = \kappa^{\omega} \leq 2^{\mathfrak{c}}$ [36]. Finally, it is well-understood which Abelian groups admit compact group topologies. Essentially these are the *only* known results in the literature about the existence of countably compact group topologies on groups of cardinality greater than $\mathfrak{c}$ (even without the additional requirement of hereditary separability).

While the algebraic description of Abelian groups admitting either a compact or a pseudocompact group topology can be carried out without any additional set-theoretic assumptions beyond ZFC, all known results about countably compact topologizations described above have either been obtained by means of some additional set-theoretic axioms (usually Continuum Hypothesis CH or versions of Martin's Axiom MA) or their consistency has been proved by forcing. Even the fundamental question (raised in [52]) as to whether the free Abelian group of size $\mathfrak{c}$ admits a countably compact group topology is still open in ZFC. (Recall that no free Abelian group admits a compact group topology.)

It seems worth noting a peculiar difference between compact and countably compact topologizations of Abelian groups. In the compact case the sufficiency of the algebraic conditions is relatively easy to prove, whereas their necessity is much harder to establish (see the proof of Theorem 12.2). In the countably compact case the necessity of **PS** and **CC** is immediate (see Lemma 2.5), while the sufficiency is rather complicated and at the present stage requires additional set-theoretic assumptions.

1.3. van Douwen's problem: Is $|G| = |G|^{\omega}$ for a countably compact group G? It is well-known that $|G| = 2^{w(G)}$ for an infinite compact group G, where $w(G)$ is the weight of G [34]. In particular, the cardinality $|G|$ of an infinite compact group G satisfies the equation $|G| = |G|^{\omega}$. This motivated van Douwen to ask in [20] the following natural question: Does $|G| = |G|^{\omega}$, or at least $\mathrm{cf}(|G|) > \omega$, hold for every infinite topological group (or homogeneous space) G which is *countably* compact?

In the same paper [20] van Douwen proved that, under the Generalized Continuum Hypothesis GCH, every infinite pseudocompact homogeneous space G satisfies $|G| = |G|^{\omega}$. In particular, a strong positive answer (with countable compactness weakened to pseudocompactness, and "topological group" weakened to "homogeneous space") to van Douwen's problem is consistent with ZFC. A first consistent counter-example to van Douwen's question was recently announced by Tomita [58] who used forcing to construct a model of ZFC in which every Boolean group of size κ has a countably compact group topology provided that $\mathfrak{c} \leq \kappa \leq 2^{\mathfrak{c}}$ [58, Theorem 2.2]. Here $2^{\mathfrak{c}}$ can be made "arbitrary large" so that, for any given ordinal $\sigma \geq 1$ chosen in advance, one can arrange that $\mathfrak{c} \leq \aleph_{\sigma} \leq 2^{\mathfrak{c}}$ (in particular, $\aleph_{\omega}$ can be included in the interval between $\mathfrak{c}$ and $2^{\mathfrak{c}}$).

In our Corollary 2.23 we push Tomita's negative solution to van Douwen's question to the extreme limit by demonstrating that, in a sense, the cofinality of $|G|$ for a countably compact Abelian group G is completely irrelevant: For every ordinal $\sigma \geq 1$ it is consistent with ZFC that *every* Abelian group G of size $\aleph_\sigma$ admits a countably compact group topology provided that G satisfies **PS** and **CC**, two necessary conditions for the existence of such a topology on G (see Definition 2.3 and Lemma 2.5(ii)).

2. Main results

The major achievement of this paper is a forcing construction of a (class of) special model(s) of ZFC in which Abelian groups of size at most $2^{\mathfrak{c}}$ admit hereditarily separable group topologies with various compactness-like properties and without non-trivial convergent sequences. Let us produce an outline of our construction.

In Section 6 we define, for every cardinal $\kappa \geq \omega_2$, a forcing notion $(\mathbb{P}_\kappa, \leq)$ that depends only on this cardinal κ (thereby justifying the notation $\mathbb{P}_\kappa$). Let M_κ be an arbitrary countable transitive model of ZFC satisfying $\kappa \in M_\kappa$, $(\mathbb{P}_\kappa, \leq) \in M_\kappa$, $\mathfrak{c} = \omega_1$ and $2^{\omega_1} = \kappa$. If $\mathbb{G} \subseteq \mathbb{P}_\kappa$ is a $\mathbb{P}_\kappa$-generic set over M_κ, then the generic extension $M_\kappa[\mathbb{G}]$ has the same cardinals as M_κ and the equalities $\omega_1 = \mathfrak{c}$ and $2^{\omega_1} = 2^{\mathfrak{c}} = \kappa$ hold in $M_\kappa[\mathbb{G}]$ (see Section 7). Since the original cardinal κ can be taken to be "arbitrarily large", the power $2^{\mathfrak{c}}$ of the continuum $\mathfrak{c}$ in the generic extension $M_\kappa[\mathbb{G}]$ can also be made "arbitrarily large".

In all results below, $M_\kappa[\mathbb{G}]$ will always denote the generic extension described above.

Our first main result shows that, at least consistently, the inequality $|G| \leq 2^{\mathfrak{c}}$ is the only necessary condition for the existence of a hereditarily separable group topology on an Abelian group:

Theorem 2.1. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*

(i) G admits a separable group topology,

(ii) G admits a hereditarily separable group topology,

(iii) G admits a hereditarily separable precompact group topology without non-trivial convergent sequences, and

(iv) $|G| \leq 2^{\mathfrak{c}}$.

Recall that Todorčević constructed a model of ZFC in which S-spaces do not exist ([54], see also [55]). Things change dramatically in this model:

Theorem 2.2. *In any model of ZFC in which there are no S-spaces the following conditions are equivalent for any Abelian group G:*

(i) G admits a hereditarily separable group topology,

(ii) G admits a separable metric precompact group topology, and

(iii) $|G| \leq \mathfrak{c}$.

We would like to emphasize that there are absolutely no algebraic restrictions (except natural restriction of commutativity) on the group G in the above two theorems. Algebraic constraints become more prominent when one adds some compactness condition to the mix.

Let G be an Abelian group. As usual $r(G)$ denotes the *free rank* of G. For every natural number $n \geq 1$ define $G[n] = \{g \in G : ng = 0\}$ and $nG = \{ng : g \in G\}$. Recall that G is:

torsion provided that $G = \bigcup\{G[n] : n \in \omega \setminus \{0\}\}$,
bounded torsion if $G = G[n]$ for some $n \in \omega \setminus \{0\}$,
torsion-free if $G[n] = \{0\}$ for every $n \in \omega \setminus \{0\}$, and
divisible if $mG = G$ for each $m \in \omega \setminus \{0\}$.

We will now introduce three algebraic conditions that will play a prominent role throughout this paper.

Definition 2.3. For an Abelian group G, define the following three conditions:
PS: Either $r(G) \geq \mathfrak{c}$ or G is a bounded torsion group.
CC: For every pair of integers $n \geq 1$ and $m \geq 1$ the group $mG[n]$ is either finite or has size at least $\mathfrak{c}$.
tCC: If G is torsion, then **CC** holds.

Our next lemma, despite its simplicity, is quite helpful for better understanding of these conditions:

Lemma 2.4. *Let G be an Abelian group.*
(i) If G is torsion, then G satisfies **PS** *if and only if G is a bounded torsion group.*
(ii) If G is a torsion-free group, then G satisfies **PS** *if and only if $|G| \geq \mathfrak{c}$.*
(iii) If G is a torsion-free group, then G satisfies **CC**.
(iv) **CC** *for G implies* **tCC**.
(v) If G is not torsion, then G satisfies **tCC**.
(vi) If G is torsion and satisfies **tCC**, *then G satisfies* **CC** *as well.*

Proof. To prove (i) note that $r(G) = 0 < \mathfrak{c}$ if G is torsion.
(ii) If G is a torsion-free group, then condition **PS** for G becomes equivalent to $r(G) \geq \mathfrak{c}$, and the latter condition is known to be equivalent to $|G| \geq \mathfrak{c}$.
(iii) Assume that G is torsion-free. Let $n \geq 1$ and $m \geq 1$ be natural numbers. Then $G[n] = \{0\}$ and hence $mG[n] = \{0\}$ is finite. Therefore **CC** holds.
Items (iv), (v) and (vi) are trivial. □

Condition **PS** is known to be necessary for the existence of a pseudocompact group topology on an Abelian group G, thereby justifying its name (**PS** stands for "pseudocompact"). To the best of the author's knowledge, this fact has been announced without proof in [4, Remark 2.17] and [12, Proposition 3.3], and has appeared in print with full proof in [14, Theorem 3.8].

It can be easily seen that condition **CC** is necessary for the existence of a countably compact group topology on an Abelian group G, thereby justifying its name (**CC** stands for "countably compact"). Indeed, if G is a countably compact group, then the set $G[n] = \{g \in G : ng = 0\}$ must be closed in G, and thus $G[n]$ is countably compact in the subspace topology induced on $G[n]$ from G. Furthermore, the map which sends $g \in G[n]$ to $mg \in mG[n]$ is continuous, and so $mG[n]$ must be countably compact (in the subspace topology). It remains only to note that an infinite countably compact group has size at least $\mathfrak{c}$ [20, Proposition 1.3 (a)]. In the particular case when an Abelian group G has size $\mathfrak{c}$, the fact that **CC** is a necessary condition for the existence of a countably compact group topology on G has been proved in [18].

Condition **CC** has essentially appeared for the first time in [12] where it was proved that **CC** is necessary for the existence of a pseudocompact group topology on a *torsion* Abelian group.[3] Since **CC** and **tCC** are equivalent for torsion groups by items (v) and (vii) of Lemma 2.4, it follows that **tCC** is a necessary condition for the existence of a pseudocompact group topology on a torsion group, thereby justifying our choice of terminology (**tCC** stands for "torsion **CC**"). Since **tCC** trivially holds for non-torsion groups (see item (vi) of Lemma 2.4), we conclude that **tCC** is a necessary condition for the existence of a pseudocompact group topology on an Abelian group G.

We can now summarize the discussion above in a convenient lemma:

Lemma 2.5. *(i) A pseudocompact Abelian group G satisfies* **PS** *and* **tCC**.
(ii) A countably compact Abelian group G satisfies **PS** *and* **CC**.

In the "opposite direction", it is known that the combination of **PS** and **tCC** is sufficient for the existence of a pseudocompact group topology on an Abelian group G of size at most $2^{\mathfrak{c}}$ ([12]; see also [14]) and, under Martin's Axiom MA, the combination of **PS** and **CC** is sufficient for the existence of a countably compact group topology on an Abelian group G of size at most $\mathfrak{c}$ [18].

In our next "twin" theorems we establish that these pairs of conditions are, consistently, also sufficient for the existence of a *hereditarily separable* pseudocompact and countably compact group topology on a group G of size at most $2^{\mathfrak{c}}$.

Theorem 2.6. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*

(i) G admits a separable pseudocompact group topology,

(ii) G admits a hereditarily separable pseudocompact group topology,

(iii) G admits a hereditarily separable pseudocompact group topology without non-trivial convergent sequences, and

[3] Furthermore, it is proved in [12] that **CC** is also a sufficient condition for the existence of a pseudocompact group topology on a *bounded* torsion Abelian group of size at most $2^{\mathfrak{c}}$. See also the proof of Theorem 2.22.

(iv) $|G| \leq 2^{\mathfrak{c}}$ *and* G *satisfies both* **PS** *and* **tCC**.

The equivalence of items (i) and (iv) in the above theorem holds in ZFC, see Theorem 13.2.

Theorem 2.7. *In* $M_\kappa[\mathbb{G}]$, *the following conditions are equivalent for any Abelian group* G*:*

(i) G *admits a separable countably compact group topology,*

(ii) G *admits a hereditarily separable countably compact group topology,*

(iii) G *admits a hereditarily separable countably compact group topology without non-trivial convergent sequences, and*

(iv) $|G| \leq 2^{\mathfrak{c}}$ *and* G *satisfies both* **PS** *and* **CC**.

Theorem 2.7 recovers (and greatly extends) the main result of Dikranjan and Tkachenko [18]: It is consistent with ZFC that an Abelian group G of size at most $\mathfrak{c}$ has a countably compact group topology if and only if G satisfies both **PS** and **CC**.

Things become "essentially trivial" in Todorčević's model of ZFC without S-spaces:

Theorem 2.8. *In any model of ZFC in which there are no* S*-spaces the following conditions are equivalent for any Abelian group* G*:*

(i) G *admits a hereditarily separable pseudocompact group topology,*

(ii) G *admits a hereditarily separable countably compact group topology, and*

(iii) G *admits a compact metric group topology.*

We refer the reader to Theorem 12.2 for the complete algebraic description of Abelian groups G that admit a compact metric group topology. This algebraic description can be added as an extra item to Theorem 2.8.

Let G be any Abelian group such that $\mathfrak{c} < |G| \leq 2^{\mathfrak{c}}$. Since compact metric spaces have size at most $\mathfrak{c}$, our previous theorem implies that, consistently, G does not admit a hereditarily separable pseudocompact group topology. On the other hand, if one additionally assumes that G satisfies both **PS** and **CC**, then G admits a hereditarily separable countably compact group topology in the model $M_\kappa[\mathbb{G}]$ (Theorem 2.7). In particular, we conclude that *the existence of a hereditarily separable pseudocompact (or countably compact) group topology on the free Abelian group of size* $2^{\mathfrak{c}}$ *is both consistent with and independent of ZFC*. An example of an Abelian group of size $\mathfrak{c}$ with similar properties is much harder to obtain. We will exhibit such a group in Example 12.4.

We will now look at what our Theorems 2.6 and 2.7 say for four particular important subclasses of Abelian groups: torsion groups, non-torsion groups, torsion-free groups, and divisible groups.

Corollary 2.9. *In* $M_\kappa[\mathbb{G}]$, *the following conditions are equivalent for any torsion Abelian group* G*:*

(i) G admits a separable pseudocompact group topology,

(ii) G admits a hereditarily separable countably compact group topology without non-trivial convergent sequences, and

(iii) $|G| \leq 2^{\mathfrak{c}}$ and G is a bounded torsion group satisfying **CC**.

Proof. Let G be a torsion Abelian group. According to Lemma 2.4(i), a bounded torsion group satisfies **PS**, so (iii) implies (ii) by Theorem 2.7. The implication (ii) $\rightarrow$ (i) is trivial. To see that (i) $\rightarrow$ (iii), note that $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **tCC** by Lemma 2.5(i). Since G is torsion, Lemma 2.4(i) yields that G is a bounded torsion group, while Lemma 2.4(vi) implies that G satisfies **CC**. □

The following particular case of the above corollary seems to be worth mentioning:

Corollary 2.10. *In $M_\kappa[\mathbb{G}]$, for every prime number p, each natural number $n \geq 1$ and every infinite cardinal τ, the following conditions are equivalent:*

(i) $\mathbb{Z}(p^n)^{(\tau)}$ admits a separable pseudocompact group topology,

(ii) $\mathbb{Z}(p^n)^{(\tau)}$ admits a hereditarily separable countably compact group topology without non-trivial convergent sequences, and

(iii) $\mathfrak{c} \leq \tau \leq 2^{\mathfrak{c}}$.

Proof. For the group $\mathbb{Z}(p^n)^{(\tau)}$, condition **CC** is equivalent to "τ is either finite or $\tau \geq \mathfrak{c}$", and the result follows from Corollary 2.9. □

Since torsion pseudocompact groups are always zero-dimensional [10], the assumption that G is non-torsion is necessary in the next two theorems.

Theorem 2.11. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any non-torsion Abelian group G:*

(i) G admits a separable pseudocompact group topology,

(ii) G admits a hereditarily separable connected and locally connected pseudocompact group topology without non-trivial convergent sequences, and

(iii) $|G| \leq 2^{\mathfrak{c}}$ and G satisfies **PS**.

Theorem 2.12. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any non-torsion Abelian group G:*

(i) G admits a separable countably compact group topology,

(ii) G admits a hereditarily separable connected and locally connected countably compact group topology without non-trivial convergent sequences, and

(iii) $|G| \leq 2^{\mathfrak{c}}$ and G satisfies **PS** *and* **CC**.

In the case of torsion-free groups things become very transparent, as algebraic restraints disappear again:

Corollary 2.13. *In $M_\kappa[\mathbb{G}]$, the following conditions are equivalent for any torsion-free Abelian group G:*

(i) G admits a separable pseudocompact group topology,

(ii) G admits a hereditarily separable countably compact connected and locally connected group topology without non-trivial convergent sequences, and
(iii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$.

Proof. Let G be a torsion-free Abelian group. According to item (iii) of Lemma 2.4, condition **PS** for G is equivalent to $|G| \geq \mathfrak{c}$, while items (iv) and (v) of the same lemma imply that both conditions **CC** and **tCC** hold for G. It remains only to plug these facts into Theorems 2.11 and 2.12. □

We note that even a very particular case of our Corollary 2.13 constitutes the main result of Koszmider, Tomita and Watson [36]: It is consistent with ZFC that for every cardinal τ such that $\mathfrak{c} \leq \tau = \tau^{\omega} \leq 2^{\mathfrak{c}}$ the free Abelian group of size τ admits a countably compact group topology without non-trivial convergent sequences. The topology constructed in [36] is not hereditarily separable, while our topology is.

As usual, for a prime number p and an Abelian group G, $r_p(G)$ denotes the *p-rank* of G. Our next theorem reduces the problem of the existence of a (hereditarily) separable countably compact group topology on a divisible Abelian group G to a simple checking of transparent conditions involving the cardinality, free rank and p-ranks of G.

Theorem 2.14. *In $M_{\kappa}[\mathbb{G}]$, the following conditions are equivalent for any non-trivial divisible Abelian group G:*
(i) G admits a separable countably compact group topology,
(ii) G admits a hereditarily separable connected and locally connected countably compact group topology without non-trivial convergent sequences,
(iii) $\mathfrak{c} \leq r(G) \leq |G| \leq 2^{\mathfrak{c}}$ *and, for every prime number p, either the p-rank $r_p(G)$ of G is finite or the inequality $r_p(G) \geq \mathfrak{c}$ holds.*

Corollary 2.15. *In $M_{\kappa}[\mathbb{G}]$, the following conditions are equivalent for any Abelian group G:*
(i) G admits a separable connected precompact group topology,
(ii) G admits a hereditarily separable connected and locally connected pseudocompact group topology without non-trivial convergent sequences.

Proof. (i) →(ii). Since G is precompact, there exists a non-trivial continuous character $\chi : G \to \mathbb{T}$. Then $\chi(G)$ is a non-trivial connected subgroup of $\mathbb{T}$, which yields $\chi(G) = \mathbb{T}$. Therefore $r(G) \geq r(\mathbb{T}) = \mathfrak{c}$. In particular, G is non-torsion and satisfies **PS**. The separability of G yields $|G| \leq 2^{\mathfrak{c}}$. Now implication (iii)→(ii) of Theorem 2.11 guarantees that G admits a hereditarily separable connected and locally connected pseudocompact group topology without non-trivial convergent sequences.
(i) →(ii) is trivial. □

Fact 1.2(i) inspired a quest for constructing compact-like group topologies without non-trivial convergent sequences, see, for example, [50, 27, 19, 38, 41, 52, 9, 58]. Our next corollary shows that, in a certain sense, one does not need to work that hard

in order to get these topologies: Indeed, at least on Abelian groups of size at most $2^{\mathfrak{c}}$, there are "plenty" of them around in the model $M_\kappa[\mathbb{G}]$:

Corollary 2.16. *In $M_\kappa[\mathbb{G}]$, let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then:*

(i) G admits a hereditarily separable precompact group topology without non-trivial convergent sequences,

(ii) if G admits a pseudocompact group topology, then G also has a hereditarily separable pseudocompact group topology without non-trivial convergent sequences,

(iii) if G admits a countably compact group topology, then G also has a hereditarily separable countably compact group topology without non-trivial convergent sequences.

Proof. Item (i) follows from the implication (iv) $\to$ (iii) of Theorem 2.1. Item (ii) follows from Lemma 2.5(i) and the implication (iv) $\to$ (iii) of Theorem 2.6. Item (iii) follows from Lemma 2.5(ii) and the implication (iv) $\to$ (iii) of Theorem 2.7. □

As a by-product of our results, we can completely describe the algebraic structure of the Abelian groups of size at most $2^{\mathfrak{c}}$ which admit, at least consistently, a countably compact group topology.

Corollary 2.17. *In $M_\kappa[\mathbb{G}]$, let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then G admits a countably compact group topology if and only if G satisfies both* **PS** *and* **CC**.

Proof. The "only if" part follows from Lemma 2.5(ii), and the "if" part follows from the implication (iv) $\to$ (iii) of Theorem 2.7. □

Corollary 2.18. *In $M_\kappa[\mathbb{G}]$, a torsion Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if G is bounded and satisfies* **CC**.

Proof. The "only if" part follows from Lemma 2.5(ii), and the "if" part follows from the implication (iii) $\to$ (ii) of Corollary 2.9. □

Corollary 2.19. *In $M_\kappa[\mathbb{G}]$, a torsion-free Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if $|G| \geq \mathfrak{c}$.*

Proof. Corollary 2.13 applies. □

Corollary 2.20. *In $M_\kappa[\mathbb{G}]$, the following two conditions are equivalent for every Abelian group G of size at most $2^{\mathfrak{c}}$ that is either torsion or torsion-free:*

(i) G admits a pseudocompact group topology, and

(ii) G admits a countably compact group topology.

Proof. Clearly (ii) implies (i). To prove the converse, assume (i). Then G satisfies **PS** and **tCC** by Lemma 2.5(i). If G is torsion, G satisfies **CC** by item (vii) of Lemma 2.4. If G is torsion-free, then G satisfies **CC** by item (iv) of Lemma 2.5. Since $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **CC**, Theorem 2.7 now yields that G has a countably compact group topology. □

Corollary 2.21. *In $M_\kappa[\mathbb{G}]$, a divisible Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if $r(G) \geq \mathfrak{c}$ and, for every prime number p, either the p-rank $r_p(G)$ of G is finite or the inequality $r_p(G) \geq \mathfrak{c}$ holds.*

Proof. This immediately follows from Theorem 2.14. □

The counterpart of Corollary 2.17 for pseudocompact group topologies can be proved in ZFC.

Theorem 2.22. *Let G be an Abelian group of size at most $2^{\mathfrak{c}}$. Then G admits a pseudocompact group topology if and only if G satisfies both* **PS** *and* **tCC**.

We will now exhibit an application of Theorem 2.7 to van Douwen's problem, see Subsection 1.3. Our next corollary demonstrates that, contrary to van Douwen's belief, it is consistent with ZFC that there is nothing exceptional about Abelian groups whose size has countable cofinality, such as $\aleph_\omega$, $\aleph_{\omega+\omega}$, $\aleph_{\omega+\omega+\omega}$ etc., from the point of view of the existence of countably compact group topologies.

Corollary 2.23. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC and $\mathfrak{c} = \omega_1$ that every Abelian group of size $\aleph_\sigma$ satisfying conditions* **PS** *and* **CC** *admits a (hereditarily separable) countably compact group topology (without non-trivial convergent sequences).*

Proof. "Make" κ bigger than $\aleph_\sigma$. Then, in $M_\kappa[\mathbb{G}]$, $2^{\mathfrak{c}} = \kappa$ will also be bigger than $\aleph_\sigma$. Now our corollary immediately follows from the conclusion of Theorem 2.7. □

Again, things become especially transparent in both torsion and torsion-free case.

Corollary 2.24. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that every bounded torsion Abelian group of size $\aleph_\sigma$ satisfying* **CC** *admits a (hereditarily separable) countably compact group topology (without non-trivial convergent sequences).*

Proof. This follows from Corollary 2.23 because bounded torsion groups satisfy **PS** (see item (ii) of Lemma 2.4). □

Corollary 2.25. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that for every prime number p and each natural number $n \geq 1$ the group $\mathbb{Z}(p^n)^{(\aleph_\sigma)}$ admits a (hereditarily separable) countably compact group topology (without non-trivial convergent sequences).*

Proof. This follows from Corollary 2.24 since the group $\mathbb{Z}(p^n)^{(\aleph_\sigma)}$ satisfies condition **CC** because $\mathfrak{c} = \aleph_1 \leq \aleph_\sigma$. □

Even a particular case of our last corollary, with $p = 2$ and $n = 1$, implies the main result of Tomita [58]: For every ordinal $\sigma \geq 1$, it is consistent with ZFC plus $\mathfrak{c} = \omega_1$ that the Boolean group $\mathbb{Z}(2)^{(\aleph_\sigma)}$ of size $\aleph_\sigma$ can be equipped with a countably compact group topology. It is also worth mentioning that the group topology constructed in

[58] is not hereditarily separable and has non-trivial convergent sequences (because it contains a Σ-product of uncountably many compact metric groups, and it is easily seen that such a Σ-product is not separable and has an infinite compact metric subgroup).

Corollary 2.26. *For every ordinal $\sigma \geq 1$, it is consistent with ZFC that every torsion-free Abelian group of size $\aleph_\sigma$ admits a (hereditarily separable) countably compact group topology (without non-trivial convergent sequences).*

Proof. "Make" κ bigger than $\aleph_\sigma$. Then, in $M_\kappa[\mathbb{G}]$, $2^{\mathfrak{c}} = \kappa$ will also be bigger than $\aleph_\sigma$. Since $\mathfrak{c} = \aleph_1 \leq \aleph_\sigma$, Corollary 2.13 applies. □

Our results on hereditary separable topologizations allow us to make a contribution to the celebrated "S-space problem" . Scattered examples of topological groups which are S-spaces are known in the literature [27, 19, 52, 38, 44, 48, 40, 51]. Our final three theorems describe completely which Abelian groups admit group topologies (with various compactness conditions) which make them into S-spaces.

Theorem 2.27. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a group topology that makes it into an S-space,
(ii) G admits a precompact group topology that makes it into an S-space,
(iii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$.

Theorem 2.28. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a pseudocompact group topology that makes it into an S-space,
(ii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **tCC**.

Theorem 2.29. *In $M_\kappa[\mathbb{G}]$, the following are equivalent for an Abelian group G:*
(i) G admits a countably compact group topology that makes it into an S-space,
(ii) $\mathfrak{c} \leq |G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **CC**.

Since hereditarily separable (initially ω_1-)compact groups are metrizable, a (initially ω_1-)compact group cannot be an S-space.

The proofs of all our theorems can be found in Section 11.

3. Algebraic preliminaries

In the sequel $\mathbb{Z}$ denotes the group of integer numbers, $\mathbb{Q}$ denotes the group of rational numbers, $\mathbb{T}$ denotes the torus group, and $\mathbf{P}$ denotes the set of prime numbers. For a cardinal κ and a group G, $G^{(\kappa)}$ denotes the sum of κ many copies of the group G and G^κ denotes the full (direct) product of κ many copies of G.

If H is an Abelian group and $h \in H$, then $\langle\langle h \rangle\rangle = \{nh : n \in \mathbb{Z}\}$ denotes the cyclic subgroup of H generated by h.

A map $\pi : G \to H$ from an Abelian group G into an Abelian group H is called a *(group) homomorphism* provided that $\pi(x + y) = \pi(x) + \pi(y)$ and $\pi(-x) = -\pi(x)$ whenever $x, y \in G$. A homomorphism $\pi : G \to H$ is a *monomorphism* if $\{x \in G : \pi(x) = 0\} = \{0\}$, i.e. if π has trivial kernel.

Lemma 3.1. *Let H be be Abelian group, H_0, H_1 its subgroups, and $\pi_i : H_i \to A$ for each $i = 0, 1$ a group homomorphism into an Abelian group A. If $\pi_0 \restriction_{H_0 \cap H_1} = \pi_1 \restriction_{H_0 \cap H_1}$, then there exists a group homomorphism $\pi : H_0 + H_1 \to A$ such that $\pi \restriction_{H_i} = \pi_i$ for every $i = 0, 1$.*

Proof. Assume that $x_0 + x_1 = x_0' + x_1'$, where $x_i, x_i' \in H_i$ for $i = 0, 1$. Then $x_0 - x_0' = x_1' - x_1 \in H_0 \cap H_1$, so by the assumption of our lemma

$$\pi_0(x_0) - \pi_0(x_0') = \pi_0(x_0 - x_0') = \pi_1(x_1' - x_1) = \pi_1(x_1') - \pi_1(x_1),$$

which yields

$$\pi_0(x_0) + \pi_1(x_1) = \pi_0(x_0') + \pi_1(x_1'). \tag{1}$$

Let $G = H_0 + H_1$. Then for every $x \in G$ there exists (not necessarily unique) representation $x = x_0 + x_1$ with $x_i \in H_i$ for $i = 0, 1$, so we can define $\pi(x) = \pi_0(x_0) + \pi_1(x_1)$. Formula (1) guarantees that π is well defined. If $i = 0, 1$ and $x \in H_i$, then $0 \in H_{1-i}$ and thus $\pi(x) = \pi_i(x) + \pi_{1-i}(0) = \pi_i(x) + 0 = \pi_i(x)$. Therefore π extends both π_0 and π_1. The straightforward check that π is a group homomorphism is left to the reader. □

The following lemma is part of algebraic folklore [24, Theorem 21.1]. We include its proof here only for the reader's convenience.

Lemma 3.2. *Let H be an Abelian group, H' its subgroup and $\pi' : H' \to T$ a group homomorphism to a divisible Abelian group T. Then there exists a group homomorphism $\pi : H \to T$ extending π'.*

Proof. If $h \in H'$, we have nothing to prove. Assume now that $h \in H \setminus H'$. Consider two cases.

Case 1: $\langle\langle h \rangle\rangle \cap H' = \{0\}$. In this case take an arbitrary group homomorphism $\pi_1 : \langle\langle h \rangle\rangle \to T$ and apply Lemma 3.1 with $H_0 = H'$, $H_1 = \langle\langle h \rangle\rangle$, $\pi_0 = \pi'$ and π_1.

Case 2: $\langle\langle h \rangle\rangle \cap H' \neq \{0\}$. In this case let $n \geq 2$ be the smallest natural number such that $nh \in H'$. Since T is divisible, there exists $t \in T$ such that $nt = \pi'(nh)$. Define homomorphism $\pi_1 : \langle\langle h \rangle\rangle \to T$ via $\pi_1(mh) = mt$ for $m \in \omega$ and note that π' and π_1 agree on the intersection $H' \cap \langle\langle h \rangle\rangle$. Therefore Lemma 3.1 can be applied to $H_0 = H'$, $H_1 = \langle\langle h \rangle\rangle$, $\pi_0 = \pi'$ and π_1. □

As a corollary we obtain a well known property of divisible subgroups found by Baer [3] (see also [24]):

Corollary 3.3. *A divisible subgroup of an Abelian group is always a direct factor.*

Proof. Let G be an Abelian group and let H be a divisible subgroup of G. According to Lemma 3.2 the identity map $id_H : H \to H$ admits an extension to a homomorphism $f : G \to H$. Then $K = \ker f$ trivially meets H and obviously $G = H + K$. Therefore, $G = H \oplus K$ holds. □

The proof of the next lemma is a standard application of Zorn's lemma:

Lemma 3.4. *Let L be a subgroup of an Abelian group G. Then there exists a maximal (under set inclusion) subgroup N of G with respect to the property $L \cap N = \{0\}$.*

Note that the subgroup N as above need not be unique.

Let us recall that a subgroup N of an Abelian group G is said to be *essential* if $H \cap N \neq \{0\}$ for every subgroup H of G with $H \neq \{0\}$. Equivalently, N is essential provided that $\langle\langle x \rangle\rangle \cap N \neq \{0\}$ for every $x \in H$ with $x \neq 0$. The importance of essentiality can be easily seen from the series of lemmas that follow.

Lemma 3.5. *Let N be a subgroup of an Abelian group G. Then N is an essential subgroup of G if and only if every homomorphism $\pi : G \to H$ to an Abelian group H is a monomorphism whenever the restriction $\pi \restriction_N \colon N \to H$ of π to N is a monomorphism.*

Proof. Assume that N is an essential subgroup of G and $\pi : G \to H$ is a homomorphism such that $\pi \restriction_N \colon N \to H$ is a monomorphism. Then the subgroup $K = \{x \in G : \pi(x) = 0\}$ of G satisfies $K \cap N = \{0\}$. By essentiality of N in H we get $K = \{0\}$, so π is a monomorphism.

Assume that the subgroup N is not essential. Then there exists a non-zero subgroup K of G such that $K \cap N = \{0\}$. Let $\pi : G \to H = G/K$ be the canonical homomorphism. Then $\pi \restriction_N \colon N \to H$ is a monomorphism, although π is not a monomorphism. □

Our next lemma provides a typical example of how essential subgroups appear naturally in algebraic proofs (compare this with Lemma 3.4).

Lemma 3.6. *Let L be a subgroup of an Abelian group G. If N is a maximal (under set inclusion) subgroup of G with respect to the property $L \cap N = \{0\}$, then $L + N = L \oplus N$ is an essential subgroup of G.*

Proof. Let H be a non-trivial subgroup of G. If $H \cap (L + N) = \{0\}$, then $L \cap (N + H) = \{0\}$, and thus $N + H = N$ by maximality of N. Therefore, $H \subseteq N$ and hence $H \cap (L+N) \supseteq H \cap N = H \neq \{0\}$. This contradiction yields $H \cap (L+N) \neq \{0\}$. □

Recall that, for a given prime number p, an Abelian group G is called a *p-group* if the period of every element of G is a power of p. The next lemma provides a typical example of an essential subgroup:

Lemma 3.7. *If H is an Abelian p-group, then $H[p]$ is an essential subgroup of H. In particular, if G is an Abelian group and p is a prime number, then $G[p]$ is an essential subgroup of $G(p^\infty) = \bigcup\{G[p^n] : n \in \omega\}$.*

Lemma 3.8. *If N is an essential subgroup of an Abelian group G, then $r(N) = r(G)$ and $r_p(N) = r_p(G)$ for every prime number p.*

Proof. Clearly $r(N) \leq r(G)$ and $r_p(N) \leq r_p(G)$ for each prime p. To prove that $r(N) = r(G)$ argue for a contradiction and assume $r(N) < r(G)$. Then there must exist an element $x \in G$ such that $\langle\langle x \rangle\rangle \cong \mathbb{Z}$ and $\langle\langle x \rangle\rangle \cap N = \{0\}$, which contradicts essentiality of N in G. Similarly, if $r_p(N) < r_p(G)$ for some prime number p, then there must exist an element $x \in G$ such that $\langle\langle x \rangle\rangle \cong \mathbb{T}[p]$ and $\langle\langle x \rangle\rangle \cap N = \{0\}$, which again contradicts essentiality of N in G. □

Our next lemma is probably known, but we include a complete proof here for readers convenience.

Lemma 3.9. *Let $\{G_i : i \in I\}$ be a family of Abelian groups. If N_i is an essential subgroup of G_i for each $i \in I$, then $\bigoplus_{i\in I} N_i$ is an essential subgroup of $\bigoplus_{i\in I} G_i$.*

Proof. First, we claim that it suffices to prove our lemma in the particular case when the set I is finite. Indeed, assuming that the finite case of our lemma has been already proved, let us prove it in the general case. Assume $g \in \bigoplus_{i\in I} G_i$ and $g \neq 0$. Pick $J \in [I]^{<\omega}$ such that $g = \sum_{j\in J} g_j$, where $g_j \in G_j$. Then $g \in \bigoplus_{j\in J} G_j$. According to the final case of our lemma, $\bigoplus_{j\in J} N_j$ is an essential subgroup of $\bigoplus_{j\in J} G_j$, which yields $\langle\langle g \rangle\rangle \cap \bigoplus_{i\in I} N_i \supseteq \langle\langle g \rangle\rangle \cap \bigoplus_{j\in J} N_j \neq \{0\}$.

It remains only to consider the case when $I = \{0, 1, \ldots, n\}$ is finite. By induction on n we will prove that $\bigoplus_{i\leq n} N_i$ is an essential subgroup of $\bigoplus_{i\leq n} G_i$.

Basis of induction. Let us prove our lemma for $n = 1$. Assume that N_0 and N_1 are essential subgroups of Abelian groups G_0 and G_1 respectively. Assume $g = g_0 + g_1 \in G_0 \oplus G_1$, where $g_i \in G_i$ for $i = 0, 1$. Suppose also that $g \neq 0$. We will consider two cases.

Case 1. $g_i = 0$ for some $i = 0, 1$. In this case $g = g_{1-i} \in G_{1-i}$, and from $g \neq 0$ and essentiality of N_{1-i} in G_{1-i}, we have $\langle\langle g \rangle\rangle \cap (N_0 \oplus N_1) \supseteq \langle\langle g \rangle\rangle \cap N_{1-i} \neq \{0\}$.

Case 2. $g_0 \neq 0$ and $g_1 \neq 0$. From $g_0 \in G_0$ and essentiality of N_0 in G_0, we get $\langle\langle g_0 \rangle\rangle \cap N_0 \neq \{0\}$. Pick $n_0 \in \mathbb{Z} \setminus \{0\}$ such that $n_0 g_0 \in N_0$ and $n_0 g_0 \neq 0$. If $n_0 g_1 = 0$, then $n_0 g = n_0(g_0 + g_1) = n_0 g_0 \neq 0$ and $n_0 g = n_0 g_0 \in N_0 \subseteq N_0 \oplus N_1$, i.e. $\langle\langle g \rangle\rangle \cap (N_0 \oplus N_1) \neq \{0\}$. Otherwise, $n_0 g_1 \neq 0$, $n_0 g_1 \in G_1$ and essentiality of N_1 in G_1 yields $\langle\langle n_0 g_1 \rangle\rangle \cap N_1 \neq \{0\}$. Therefore $n_1 n_0 g_1 \in N_1$ and $n_1 n_0 g_1 \neq 0$ for some $n_1 \in \mathbb{Z} \setminus \{0\}$. Now note that $n_1 n_0 g_0 \in n_1 N_0 \subseteq N_0$ and $n_1 n_0 g_1 \in N_1$ implies $n_1 n_0 g = n_1 n_0(g_0 + g_1) = n_1 n_0 g_0 + n_1 n_0 g_1 \in N_0 \oplus N_1$. Furthermore, $n_1 n_0 g_1 \neq 0$ implies $n_1 n_0 g = n_1 n_0 g_0 + n_1 n_0 g_1 \neq 0$ because $G_0 \oplus G_1$ is the direct sum. Therefore $\langle\langle g \rangle\rangle \cap (N_0 \oplus N_1) \neq \{0\}$.

Inductive step. Assume that $n \in \omega$, $n \geq 2$ and our lemma has been proved for all $k < n$. For $i \leq n$ let N_i be an essential subgroup of an Abelian group G_i. By our inductive assumption, $\bigoplus_{i\leq n-1} N_i$ is an essential subgroup of $\bigoplus_{i\leq n-1} G_i$. Applying the inductive assumption once again (to the case of two groups), we conclude that $(\bigoplus_{i\leq n-1} N_i) \oplus N_n$ is an essential subgroup of $(\bigoplus_{i\leq n-1} G_i) \oplus G_n$. □

Recall that a family $\mathcal{N} = \{N_i : i \in I\}$ of subgroups of G is said to be *independent*, if the subgroup N generated by $\mathcal{N}$ is their direct sum $\bigoplus_{i\in I} N_i$. In other words, if

$J \in [I]^{<\omega}$, $g_j \in N_j$ for every $j \in J$ and $\sum_{j\in J} g_j = 0$, then $g_j = 0$ for all $j \in J$. Equivalently, $\mathcal{N}$ is independent if and only if every element $g \in N$ admits a *unique* representation $g = \sum_{i\in I} g_i$ where $g_i \in N_i$ for every $i \in I$ and the set $\{i \in I : g_i \neq 0\}$ is finite.

Lemma 3.10. *Let H be an Abelian group and $\{G_i : i \in I\}$ an independent family of subgroups of an Abelian group G. For each $i \in I$ let $\pi_i : G_i \to H$ be a group homomorphism. Then:*

(i) There exists a unique group homomorphism $\pi = \bigoplus_{i\in I} \pi_i : \bigoplus_{i\in I} G_i \to H$ such that $\pi \restriction_{G_i} = \pi_i$ for $i \in I$. Moreover,

(ii) If each π_i is a monomorphism and the family $\{\pi_i(G_i) : i \in I\}$ of subgroups of H is independent, then π is also a monomorphism.

Proof. (i) This is the well known categorical characterization of the direct sum. [4]

(ii) Since each π_i is a monomorphism, there exists a unique inverse $\pi_i^{-1} : \pi_i(G_i) \to G_i$. Item (i) yields the existence of a homomorphism $\rho = \bigoplus_{i\in I} \pi_i^{-1} : \bigoplus_{i\in I} \pi_i(G_i) \to \bigoplus_{i\in I} G_i$ such that $\rho \restriction_{\pi_i(G_i)} = \pi_i^{-1}$ for $i \in I$. Obviously, $\rho \circ \pi$ is the identity map of $\bigoplus_{i\in I} G_i$, and hence π is a monomorphism. □

Lemma 3.11. *Let $\{G_i : i \in I\}$ be a family of subgroups of an Abelian group H. Assume also that N_i is an essential subgroup of G_i for $i \in I$. Then the family $\{G_i : i \in I\}$ is independent if and only if the family $\{N_i : i \in I\}$ is independent.*

Proof. The "only if" part is obvious. To prove the "if" part, assume that the family $\{N_i : i \in I\}$ is independent. By Lemma 3.10 there exists a unique group homomorphism $\pi : G = \bigoplus_{i\in I} G_i \to H$ such that $\pi \restriction_{G_i} : G_i \hookrightarrow H$ is the inclusion for $i \in I$. By the independence of the family $\{N_i : i \in I\}$, the restriction of π to the subgroup $N = \bigoplus_{i\in I} N_i$ of G is a monomorphism. According to Lemma 3.9, N is an essential subgroup of G. Therefore π is a monomorphism by Lemma 3.5. Hence the family $\{G_i : i \in I\}$ is independent. □

In view of Lemma 3.7, our next lemma is a particular case of Lemma 3.11:

Lemma 3.12. *Let $\{H_i : i \in I\}$ be a family of non-zero p-subgroups of an Abelian group G. If the family $\{H_i[p] : i \in I\}$ is independent, then also the family $\{H_i : i \in I\}$ is independent.*

Lemma 3.13. *Let p be a prime number, $n \in \omega \setminus \{0\}$, G an Abelian group and V a subgroup of $p^{n-1}G[p^n]$ of size ω_1, then there exists a subgroup N of G with $N \cong \mathbb{T}[p^n]^{(\omega_1)}$ and $N[p] = V$.*

Proof. Pick a base $\{x_i\}_{i\in I}$ of V of size ω_1. Since $x_i \in p^{n-1}G[p^n]$, we can find $y_i \in G[p^n]$ such that $x_i = p^{n-1}y_i$. Then $\langle\langle y_i \rangle\rangle \cong \mathbb{T}[p^n]$ for every $i \in I$. Moreover,

[4] For reader's convenience we give a proof here. Each $g \in \bigoplus_{i\in I} G_i$ has a unique representation $g = \sum_{i\in I} g_i$ where $g_i \in N_i$ for $i \in I$ and the set $\{i \in I : g_i \neq 0\}$ is finite. We define $\pi(g) = \pi(\sum_{i\in I} g_i) = \sum_{i\in I} \pi_i(g_i)$. It is easy to check that π satisfies the requirements of item (i).

$\langle\!\langle y_i \rangle\!\rangle[p] = \langle\!\langle x_i \rangle\!\rangle$, for $i \in I$, form an independent family. Hence by Lemma 3.12 also $\{\langle\!\langle y_i \rangle\!\rangle : i \in I\}$ is an independent family. Therefore, the elements $\{y_j : j \in J\}$ generate a subgroup N of G isomorphic to $\mathbb{T}[p^n]^{(\omega_1)}$. Obviously, $N[p] = V$. □

Lemma 3.14. *Let $\{S_n : n \in \omega\}$ be a sequence of subgroups of an Abelian group G with $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq S_{n+1} \supseteq \ldots$. Assume also that $\{V_n : n \in \omega\}$ is a sequence of subgroups of G such that $V_n \subseteq S_n$ and $V_n \cap S_{n+1} = \emptyset$ for every $n \in \omega$. Then the family $\{V_n : n \in \omega\}$ is independent.*

Proof. Assume the contrary, and fix $J \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $g_j \in V_j$ for every $j \in \omega$ such that $\sum_{j\in J} g_j = 0$ but $g_j \neq 0$ for all $j \in J$. Let n be the smallest integer in J. Then $j \in J \setminus \{n\}$ implies $n < j$ and thus $g_j \in V_j \subseteq S_j \subseteq S_{n+1}$. Therefore, $\sum_{j\in J\setminus\{n\}} g_j \in S_{n+1}$. From $-g_n = \sum_{j\in J\setminus\{n\}} g_j \in V_n \cap S_{n+1} = \{0\}$ we get $g_n = 0$, a contradiction. □

For an Abelian group G we denote by $Soc(G)$ the sum of all subgroups $G[p]$ of G when p runs over the set $\mathbf{P}$ of all prime numbers. We show now that this sum is direct.

Lemma 3.15. *Let G be an Abelian group, $t(G)$ its torsion subgroup and N the maximal (with respect to set inclusion) subgroup of G satisfying $t(G) \cap N = \{0\}$.*[5] *Then:*

(i) the family $\{G[p] : p \in \mathbf{P}\} \cup \{N\}$ is independent, and
(ii) $Soc(G) = \bigoplus_{p\in\mathbf{P}} G[p]$ is an essential subgroup of G.

Proof. Item (i) of our lemma is trivial. Let us check (ii). Lemma 3.6 yields that $t(G) \oplus N$ is an essential subgroup of G. In view of Lemma 3.9 and transitivity of essentiality, it remains only to prove that $Soc(G)$ is an essential subgroup of $t(G)$. Recall that, for $p \in \mathbf{P}$, $G(p^\infty) = \bigcup\{G[p^n] : n \in \omega \setminus \{0\}\}$ denotes the largest p-subgroup of G. Clearly the family $\{G(p^\infty) : p \in \mathbf{P}\}$ is independent, and so $t(G) = \bigoplus\{G(p^\infty) : p \in \mathbf{P}\}$. From Lemmas 3.7 and 3.9 one concludes that $Soc(G)$ is an essential subgroup of $t(G)$. □

In the next lemma we show that the cardinal invariants $r(-)$ and $r_p(-)$ alone can determine a lot about when an Abelian group can be embedded into another Abelian group.

Lemma 3.16. *Suppose that G and H are Abelian groups such that $r(G) \leq r(H)$ and $r_p(G) \leq r_p(H)$ for all $p \in \mathbf{P}$. Then there is a monomorphism $\pi : G' \to H$ defined on an essential subgroup G' of G Moreover, if H is assumed to be divisible, then G' can be chosen to coincide with G.*

Proof. Use Lemma 3.4 twice to pick a maximal subgroup N_G of G with $t(G) \cap N_G = \{0\}$ and a maximal subgroup N_H of H with $t(H) \cap N_H = \{0\}$. By Lemma 3.15(i)

[5] Which exists by Lemma 3.4.

$\{G[p] : p \in \mathbf{P}\} \cup \{N_G\}$ is an independent family in G and $\{H[p] : p \in \mathbf{P}\} \cup \{N_H\}$ is an independent family in H. Combining Lemmas 3.15(ii) and 3.8 with the assumption of our lemma gives

$$r(N_G) = r(Soc(G) \oplus N_G) = r(G) \leq r(H) = r(Soc(H) \oplus N_H) = r(N_H),$$

and thus there is a monomorphism $\pi_0 : L_G \to N_H$ defined on an essential subgroup $L_G \cong \mathbb{Z}^{(r(G))}$ of N_G. Similarly, for $p \in \mathbf{P}$, we have

$$r_p(G[p]) = r_p(Soc(G) \oplus N_G) = r_p(G) \leq r_p(H) = r_p(Soc(H) \oplus N_H) = r_p(H[p]),$$

and so there is a monomorphism $\pi_p : G[p] \to H[p]$. Since L_G is a subgroup of N_G, $\{G[p] : p \in \mathbf{P}\} \cup \{L_G\}$ is an independent family in G by Lemma 3.15(i). Since $\pi_p(G[p]) \subseteq H[p]$ for $p \in \mathbf{P}$ and $\pi_0(L_G) \subseteq N_H$, reference to Lemma 3.15(i) once again gives us that the family $\{\pi_p(G[p]) : p \in \mathbf{P}\} \cup \{\pi_0(L_G)\}$ of subgroups of H is independent. According to Lemma 3.10, there exists a unique monomorphism $\pi : G' \to H$ such that $\pi \restriction_{G[p]} = \pi_p$ for $p \in \mathbf{P}$ and $\pi \restriction_{L_G} = \pi_0$, where $G' = Soc(G) \oplus L_G$. Since L_G is essential subgroup of N_G, G' is an essential subgroup of $Soc(G) \oplus N_G$ (Lemma 3.9), and the latter subgroup is essential in G by Lemma 3.15(ii). Since essentiality is transitive, G' is essential in G.

Suppose now, in addition, that H is divisible. Lemma 3.2 allows us to find a homomorphism $\varphi : G \to H$ extending π. Since the restriction $\varphi \restriction_{G'} = \pi$ of φ to the essential subgroup G' of G is a monomorphism, φ itself is a monomorphism (Lemma 3.5). □

Lemma 3.17. *Let G and H be Abelian groups such that $|G| \leq r(H)$ and $|G| \leq r_p(H)$ for each $p \in \mathbf{P}$. Suppose also that G' a subgroup of G such that $r(G') < r(H)$ and $r_p(G') < r_p(H)$ for all $p \in \mathbf{P}$. If H is divisible, then for every monomorphism $\phi : G' \to H$ there exists a monomorphism $\varphi : G \to H$ such that $\varphi \restriction_{G'} = \phi$.*

Proof. Define $H' = \phi(G')$. Use Lemma 3.4 twice to pick a maximal subgroup N_G of G with $G' \cap N_G = \{0\}$ and a maximal subgroup N_H of H with $H' \cap N_H = \{0\}$. Since $G' + N_G = G' \oplus N_G$ is an essential subgroup of G by Lemma 3.6, Lemma 3.8 yields

$$r(G) = r(G' \oplus N_G) = r(G') + r(N_G). \tag{2}$$

Similarly, since $H' + N_H = H' \oplus N_H$ is an essential subgroup of H by Lemma 3.6, Lemma 3.8 yields

$$r(H) = r(H' \oplus N_H) = r(H') + r(N_H) = r(G') + r(N_H), \tag{3}$$

where in the last equation we used the fact that $H' = \phi(G')$ and ϕ is a monomorphism. Therefore,

$$r(G') + r(N_G) = r(G) \leq |G| \leq r(H) = r(G') + r(N_H). \tag{4}$$

If $r(H)$ is finite, from (4) one immediately gets $r(N_G) \leq r(N_H)$. Otherwise, the hypothesis $r(G') < r(H)$ and (4) yield $r(N_G) \leq r(N_H)$.

Similarly, for a given $p \in \mathbf{P}$, again by essentiality and Lemma 3.8 we obtain the following p-versions of (2) and (3), respectively:

$$r_p(G) = r_p(G' \oplus N_G) = r_p(G') + r_p(N_G), \tag{5}$$

$$r_p(H) = r + p(H' \oplus N_H) = r_p(H') + r_p(N_H) = r_p(G') + r_p(N_H). \tag{6}$$

From (5) and (6) one gets

$$r_p(G') + r_p(N_G) = r_p(G) \leq |G| \leq r_p(H) = r_p(G') + r_p(N_H). \tag{7}$$

If $r_p(H)$ is finite, (7) implies $r_p(N_G) \leq r_p(N_H)$. Otherwise, the hypothesis $r_p(G') < r_p(H)$ and (7) yield $r_p(N_G) \leq r_p(N_H)$.

Since $r(N_G) \leq r(N_H)$ and $r_p(N_G) \leq r_p(N_H)$ for every $p \in \mathbf{P}$, there exist an essential subgroup N'_G of N_G and a monomorphism $\psi : N'_G \to N_H$ (Lemma 3.16). By our choice of N_G and N_H, it follows that $\{G', N_G\}$ is an independent family in G and $\{H', N_H\}$ is an independent family in H. Then the sum $\phi \oplus \psi : G' \oplus N'_G \to H' \oplus N_H \subseteq H$ of monomorphisms ϕ and ψ is again a monomorphism by Lemma 3.10. Since H is divisible, we can use Lemma 3.2 to find a homomorphism $\varphi : G \to H$ extending $\phi \oplus \psi$. Since N'_G is an essential subgroup of N_G, $G' \oplus N'_G$ is an essential subgroup of $G' \oplus N_G$ (Lemma 3.9), and the latter is an essential subgroup of G by Lemma 3.6. Hence $G' \oplus N'_G$ is an essential subgroup of G by transitivity of essentiality. Finally, Lemma 3.5 guarantees that φ is a monomorphism. □

Lemma 3.18. *Let G and H be Abelian groups such that $|G| \leq r(H)$ and $|G| \leq r_p(H)$ for each $p \in \mathbf{P}$. If H is divisible, then there exists a monomorphism $\phi : G \to H$. In particular, G is algebraically isomorphic to a subgroup of H.*

Proof. Apply Lemma 3.17 with $G' = \{0\}$ and ϕ' the trivial monomorphism. □

4. ALMOST n-TORSION SETS

For a subset E of an Abelian group G and $n \in \omega$ we define $nE = \{nx : x \in E\}$.

We will say that $d \in \omega$ is a *proper divisor of* $n \in \omega$ provided that $d \notin \{0, n\}$ and $dm = n$ for some $m \in \omega$. Note that, according to our definition, each $d \in \omega \setminus \{0\}$ is a proper divisor of 0.

Definition 4.1. Let H be an Abelian group. For a given $n \in \omega$ we will say that $E \subseteq [H]^\omega$ is *almost n-torsion in H* if $nE = \{0\}$ and the set $\{x \in E : dx = h\}$ is finite for each $h \in H$ and every proper divisor d of n.

We note that there are no almost 1-torsion sets. When $n \geq 2$, it can be easily seen that a set E is almost n-torsion in H if and only if all but finitely many elements of E are *n-torsion*, i.e. have order n.[6] This serves as a justification for our terminology. We would like to note the following important fact:

[6]Recall that $x \in H$ has *order* n provided that $nx = 0$ but $mx \neq 0$ whenever $0 < m < n$.

Lemma 4.2. *Let E be a subset of an Abelian group H and $n \in \omega$. Then the following conditions are equivalent:*

(i) E is almost n-torsion in H,

(ii) E is almost n-torsion in the smallest subgroup $\langle\langle E \rangle\rangle$ of H that contains E.

Proof. It suffices to note that, for every $n \geq 1$, the set $\{x \in E : nx = h\}$ is empty unless $h \in \langle\langle E \rangle\rangle$. □

Our next lemma says that almost n-torsionness of a set E is an absolute property in a sense that it does not depend on the group that contains E:

Lemma 4.3. *Let H be a group, G its subgroup and $E \subseteq G$. Then E is almost n-torsion in G if and only if E is almost n-torsion in H.*

Proof. Note that $\langle\langle E \rangle\rangle \subseteq G \subseteq H$. Applying Lemma 4.2 twice, we conclude that E is almost n-torsion in H iff E is almost n-torsion in $\langle\langle E \rangle\rangle$ iff E is almost n-torsion in G. □

Lemma 4.4. *If G is an Abelian group, $m \in \omega \setminus \{0\}$, $d \in \omega \setminus \{0, 1\}$, $n = dm$ and E is an almost n-torsion subset of G, then $E' = \{mg : g \in E\}$ is an almost d-torsion subset of G.*

Proof. Note that m is a proper divisor on n. Since E is almost n-torsion, the set $\{g \in E : mg = h\}$ is finite for every $h \in G$, which implies that E' is an infinite set. Since $E \subseteq G[n]$ and $n = dm$, we have $E' \subseteq G[d]$. Suppose now that k is a proper divisor of d, i.e. $d = d'k$ with $d' \neq 1$. Then $n = dm = d'km$ and thus km is a proper divisor of n. Let $h \in G$. Since E is almost n-torsion, the set $E_h = \{g \in E : kmg = h\}$ is finite, and thus the set $\{g' \in E' : kg' = h\} \subseteq \{mg : g \in E_h\}$ must be finite as well. □

It is easy to check that our almost n-torsion sets for $n \geq 2$ coincide with n-round sets in the sense of [18, Definition 3.3], while our almost 0-torsion sets are precisely the admissible sets in the sense of [18, Definition 3.3]. Having in mind the above mentioned differences, we can borrow the following fact from [18] which we will need in the sequel.

Lemma 4.5. ([18, Lemma 3.6]) *Let S be an infinite subset of an Abelian group H. Then there exist $n \in \omega \setminus \{1\}$, an element $h \in H$ and an almost n-torsion set E in H such that $h + E \subseteq S$.*

For a (discrete) Abelian group G we use G^* to denote the group of characters equipped with the topology of pointwise convergence. That is, $G^* = \{f : G \to \mathbb{T}$ is a group homomorphism$\}$, and a base of the topology of G^* consists of the sets

$$W(h, k, n; x_0, x_1, \ldots, x_n) = \{f \in G^* : \forall i \leq n \ |f(x_i) - h(x_i)| < 1/k\},$$

where $h \in G^*$, $k, n \in \omega$ and $x_0, x_1, \dots, x_n \in G$. It is well-known that G^* is compact.[7]

Lemma 4.6. ([18, Lemma 4.2]) *Let E be an almost 0-torsion subset of an Abelian group G. Then the set $\mathcal{F}_E = \{f \in G^* : f(E)$ is dense in $\mathbb{T}\}$ is the intersection of countably many open dense subsets of G^*.*

Even though the proof of our next lemma can be extracted from the proof of [18, Lemma 3.7], we include it here for the reader's convenience.

Lemma 4.7. *For a natural number $n > 1$ let E be an almost n-torsion subset of an Abelian group G. Then for every $z \in \mathbb{T}[n]$ the set*

$$U_E^n(z) = \{f \in G^* : \exists x \in E \ f(x) = z\}$$

is open and dense in G^*.

Proof. Let us first verify that $U_E^n(z)$ is open in G^*. Indeed, let $f \in U_E^n(z)$. Then $f(x) = z$ for some $x \in E$. Observe that $V = \{f' \in G^* : |f'(x) - f(x)| < 1/n\}$ is an open subset of G^* with $f \in V$. It remains only to check that $V \subseteq U_E^n(z)$. Indeed, let $f' \in V$ be arbitrary. Then $f(x) = z$ and definition of V implies that $|f'(x) - z| < 1/n$. Since $x \in E \subseteq G[n]$, it follows that $nx = 0$ and thus $nf'(x) = f'(nx) = f'(0) = 0$, i.e. $f'(x) \in \mathbb{T}[n]$. Note also that $z \in \mathbb{T}[n]$. Since different elements of $\mathbb{T}[n]$ are at least distance $1/n$ apart, the condition $|f'(x) - z| < 1/n$ now yields $f'(x) = z$. Therefore, $f' \in U_E^n(z)$.

Let us now prove that $U_E^n(z)$ is dense in G^*. It suffices to prove that, for a given $h \in G^*$, $k, n \in \omega$ and $x_0, x_1, \dots, x_n \in G$, one has $W(h, k, n; x_0, x_1, \dots, x_n) \cap U_E^n(z) \neq \emptyset$.

Since E is almost n-torsion, the set $E \cap G[d]$ is finite for every proper divisor d of n. Therefore, there exists a finite set F such that each element of $E \setminus F$ has order n. Since $N = \langle\langle x_0 \rangle\rangle + \langle\langle x_1 \rangle\rangle + \dots + \langle\langle x_n \rangle\rangle$ is a finitely generated subgroup of G, its torsion part $t(N)$ is finite. We claim that there exists a non-zero element $x \in E \setminus F$ such that $\langle\langle x \rangle\rangle \cap N = \{0\}$. Suppose the contrary, i.e. $\langle\langle x \rangle\rangle \cap N \neq \{0\}$ for every $x \in E \setminus F$. Then for every $x \in E \setminus F$ one can find a proper divisor d_x of n such that $d_x x \in N$. Note that, in fact, $d_x x \in t(N)$ for every $x \in E \setminus F$. Since both $t(N)$ and the set of proper divisors of n are finite and the set $E \setminus F$ is infinite, there exist an infinite $E' \subseteq E$, a proper divisor d of n and $a \in t(N)$ such that $d_x = d$ and $d_x x = dx = a$ for all $x \in E'$. This contradicts the fact that E is almost n-torsion.

Pick now $x \in E \setminus F$ with $\langle\langle x \rangle\rangle \cap N = \{0\}$. Since $z \in \mathbb{T}[n]$ and x has order n, there exists a homomorphism $\pi_1 : \langle\langle x \rangle\rangle \to \mathbb{T}$ such that $\pi(x) = z$. We can now apply Lemma 3.1 to $H_0 = N$, $\pi_0 = h \restriction_N$, $H_1 = \langle\langle x \rangle\rangle$ and π_1 to get a homomorphism $\pi : N + \langle\langle x \rangle\rangle \to \mathbb{T}$ such that $\pi(x_i) = \pi \restriction_N (x_i) = h \restriction_N (x_i) = h(x_i)$ for all $i \leq n$

[7]Indeed, the usual Tychonoff product topology on $\mathbb{T}^{G^*}$ is compact by the celebrated Tychonoff compactness theorem. Clearly, G^* is a subspace of $\mathbb{T}^{G^*}$. Now it can be easily seen that G^* is *closed* in $\mathbb{T}^{G^*}$ since the failure of an arbitrary map $f : G \to \mathbb{T}$ to be a group homomorphism is witnessed by a subset of G of size at most 2.

and $\pi(x) = \pi_1(x) = z$. Let $f : G \to \mathbb{T}$ be any group homomorphism extending π. Then $f(x_i) = h(x_i)$ for all $i \leq n$, which yields $f \in W(h, k, n; x_0, x_1, \ldots, x_n)$. Finally, $f(x) = \pi(x) = z$ with $x \in E$, which implies $f \in U^n_E(z)$. □

Lemma 4.8. *For a natural number $n > 1$ let E be an almost n-torsion subset of an Abelian group G. Then the set*

$$\mathcal{F}^n_E = \{f \in G^* : \forall z \in \mathbb{T}[n] \ \{x \in E : f(x) = z\} \text{ is infinite}\}$$

contains an intersection of countably many open dense subsets of G^*.

Proof. Partition E into countably many pairwise disjoint infinite sets E_m, and define $B = \bigcap\{U^n_{E_m}(z) : m \in \omega, z \in \mathbb{T}[n]\}$. Since each E_m is almost n-torsion (being an infinite subset of an almost n-torsion set), each set $U^n_{E_m}(z)$ is open and dense in G^* by Lemma 4.7. The inclusion $B \subseteq \mathcal{F}^n_E$ is immediate from the fact that $\{E_m : m \in \omega\}$ is a partition of E. □

It will be convenient for us to define $\mathbb{T}[0] = \bigcup\{\mathbb{T}[n] : n \in \omega \setminus \{0\}\}$ so that $\mathbb{T}[0]$ becomes exactly the subgroup of torsion elements of $\mathbb{T}$. It is well-known that $\mathbb{T}[0]$ is dense in $\mathbb{T}$. With this notation in mind, we are now ready to prove the following key lemma which will be essential for our forcing construction:

Lemma 4.9. *Suppose that G be an Abelian group, $g \in G$, $g \neq 0$ and $\mathcal{E} \subseteq [G]^\omega$ is a countable family. Then there exists a group homomorphism $\varpi : G \to \mathbb{T}$ such that:*

(i) $\varpi(g) \neq 0$,

(ii) if $n \in \omega$, $z \in \mathbb{T}[n]$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}$ is almost n-torsion in G, then the set $\{h \in E : |\varpi(h) - z| < 1/k\}$ is infinite.

Proof. For every $n \in \omega$ let $\mathcal{E}_n = \{E \in \mathcal{E} : E \text{ is almost } n\text{-torsion}\}$. Since $|\mathcal{E}_n| \leq |\mathcal{E}| \leq \omega$ for each $n \in \omega$, Lemmas 4.5 and 4.8 allow us to conclude that there exists a countable family $\{V_n : n \in \omega\}$ of open dense subsets of G^* such that

$$B = \bigcap\{V_n : n \in \omega\} \subseteq \bigcap\{\mathcal{F}_E : E \in \mathcal{E}_0\} \cap \bigcap\{\mathcal{F}^n_E : n \in \omega \setminus \{0\}, E \in \mathcal{E}_n\}.$$

Pick arbitrarily a homomorphism $h : G \to \mathbb{T}$ such that $h(g) \neq 0$ and choose $k \in \omega \setminus \{0\}$ with $1/k < |h(g) - 0| \neq 0$. Then $W(h, k, 0; g) = \{f \in G^* : |f(g) - h(g)| < 1/k\}$ is a nonempty open set in G^* such that $f \in W(h, k, 0; g)$ implies $f(g) \neq 0$. Since G^* is compact, by the Baire category theorem, the intersection

$$W(h, k, 0; g) \cap B = W(h, k, 0; g) \cap \bigcap\{V_n : n \in \omega\}$$

must be nonempty. We now claim that any $\varpi \in W(h, k, 0; g) \cap B$ will satisfy the conclusion of our lemma. From $\varpi \in W(h, k, 0; g)$ it follows that $\varpi(g) \neq 0$, i.e. (i) holds.

Let us check (ii). Assume that $n \in \omega$, $z \in \mathbb{T}[n]$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}$ is almost n-torsion in G. We need to show that the set $\{h \in E : |\varpi(h) - z| < 1/k\}$ is infinite. We will consider two cases.

Case 1: $n = 0$. Then $E \in \mathcal{E}_0$, and from our choice of ϖ it follows that $\varpi \in \mathcal{F}_E$, i.e. the set $\varpi(E)$ is dense in $\mathbb{T}$. Since $\{y \in \mathbb{T} : |y - z| < 1/k\}$ is an open subset of $\mathbb{T}$, by denseness of $\varpi(E)$ in $\mathbb{T}$, the intersection $\{y \in \mathbb{T} : |y - z| < 1/k\} \cap \varpi(E)$ must be infinite. Then the set $\{h \in E : |\varpi(h) - z| < 1/k\}$ must be infinite as well.

Case 2: $n \geq 1$. Then $E \in \mathcal{E}_n$, and from our choice of ϖ it follows that $\varpi \in \mathcal{F}_E^n$, which implies that the set $\{x \in E : \varpi(x) = z\}$ is infinite. Since $\{x \in E : \varpi(x) = z\} \subseteq \{x \in E : |\varpi(x) - z| < 1/k\}$, the latter set must be infinite as well. □

5. Algebraic embedding property

In this section we prove an algebraic lemma needed for our forcing proof. Recall that $\mathbf{P}$ denotes the set of prime numbers.

Definition 5.1. (i) Let $K_0 = \mathbb{Z}^{(\omega_1)}$ and $K_n = \mathbb{T}[n]^{(\omega_1)}$ for every $n \in \omega \setminus \{0\}$.
(ii) For every prime p we define $K_{p^\infty} = \bigoplus_{n \in \omega} K_{p^n}$.
(iii) Define $K = K_0 \oplus \bigoplus_{p \in \mathbf{P}} K_{p^\infty}$.

Definition 5.2. Let G be an Abelian group.
(i) We define $\mathfrak{R}(G) \subseteq \omega$ by declaring $n \in \mathfrak{R}(G)$ if and only if G contains at least one almost n-torsion subset.
(ii) For a prime p let $K_p(G) = \bigoplus_{p^m \in \mathfrak{R}(G)} K_{p^m}$.
(iii) Let $K_0(G) = \{0\}$ if $r(G) = 0$, otherwise $K_0(G) = K_0$.
(iv) Define $K(G) = K_0(G) \oplus \bigoplus_{p \in \mathbf{P}} K_p(G)$.

Note that $1 \notin \mathfrak{R}(G)$ because almost 1-torsion sets do not exist. We will need "downward closedness" of the set $\mathfrak{R}(G)$:

Lemma 5.3. *If G is an Abelian group, $n \in \mathfrak{R}(G) \setminus \{0\}$ and $d \neq 1$ is a divisor of n, then $d \in \mathfrak{R}(G)$.*

Proof. Since $n \in \mathfrak{R}(G) \setminus \{0\}$, there exists an almost n-torsion set $E \in [G]^\omega$. We also have $dm = n$ with some $m \in \omega$. By Lemma 4.4, $E' = \{mg : g \in E\}$ is an almost d-torsion set in G, and hence $d \in \mathfrak{R}(G)$. □

Lemma 5.4. *If G is an Abelian group, then $K(G)$ contains an isomorphic copy of K_n for every $n \in \mathfrak{R}(G) \setminus \{0\}$.*

Proof. Since there are no almost 1-torsion sets, $1 \notin \mathfrak{R}(G)$. We can therefore assume that $n \geq 2$. Let $n = p_1^{k_1} \dots p_s^{k_s}$, where $p_1, \dots, p_s$ are distinct primes and $k_1, \dots, k_s$ are positive integers. Then $n \in \mathfrak{R}(G)$ implies that $p_i^{k_i} \in \mathfrak{R}(G)$ for every $i = 1, 2, \dots, s$ (Lemma 5.3), and so $K_{p_1^{k_1}} \oplus \dots \oplus K_{p_s^{k_s}}$ is (isomorphic to) a subgroup of $K(G)$. It remains only to note that there is a canonical isomorphism $K_n \cong K_{p_1^{k_1}} \oplus \dots \oplus K_{p_s^{k_s}}$. □

Lemma 5.5. *If G is an Abelian group satisfying* **PS**, *then $0 \in \mathfrak{R}(G)$ implies that G contains a subgroup algebraically isomorphic to $K_0(G) \cong K_0$.*

Proof. Note that a bounded torsion group cannot contain almost 0-torsion sets, so combining $0 \in \mathfrak{R}(G)$ and **PS** yields $r(G) \geq \mathfrak{c}$. Therefore G contains a subgroup algebraically isomorphic to $K_0(G) \cong K_0 = \mathbb{Z}^{(\omega_1)}$. □

Lemma 5.6. *If G is an Abelian group satisfying condition* **CC** *and p is a prime number, then $n \in \omega \setminus \{0\}$ and $p^n \in \mathfrak{R}(G)$ implies that G has a subgroup algebraically isomorphic to K_{p^n}.*

Proof. First let us settle the case $n = 1$. From $p \in \mathfrak{R}(G)$ it follows that G has an almost p-torsion set $E \subseteq G[p]$. In particular, $G[p]$ is infinite and thus $|G[p]| \geq \mathfrak{c}$ by **CC**. Hence $G[p]$ contains a subgroup algebraically isomorphic to K_p.

Assume now that $n \geq 2$. From $p^m \in \mathfrak{R}(G)$ it follows that G has an almost p^n-torsion set $E \subseteq G[p^n]$. Now Lemma 4.4 implies that $\{p^{n-1}x : x \in E\} \subseteq p^{n-1}G[p^n]$ is an almost p-torsion set. Therefore, the set $p^{n-1}G[p^n]$ is infinite, and **CC** implies that $|p^{n-1}G[p^n]| \geq \mathfrak{c}$. Let V be a subgroup of $p^{n-1}G[p^n]$ of size ω_1. Application of Lemma 3.13 now yields a subgroup N of G with $N \cong \mathbb{T}[p^n]^{(\omega_1)} \cong K_{p^n}$. □

Lemma 5.7. *If G is an Abelian group satisfying condition* **CC**, *then for every prime number p the group G contains a subgroup algebraically isomorphic to $K_p(G)$.*

Proof. Fix $p \in \mathbf{P}$ and let $\Omega_p = \{n \in \omega \setminus \{0\} : p^n \in \mathfrak{R}(G)\}$. According to Lemma 5.3, either $\Omega_p = \{1, \dots, n\}$ for some $n \in \omega$, or $\Omega_p = \omega$. The former case is trivial since in this case $K_p(G) = \bigoplus_{k=1}^n K_{p^k}$ is isomorphic to a subgroup of $K_{p^n}^n \cong K_{p^n}$ and G contains a subgroup algebraically isomorphic to K_{p^n} by Lemma 5.6 (note that $p^n \in \mathfrak{R}(G)$). So we are left with the latter case $\Omega_p = \omega$, i.e. $K_p(G) = K_{p^\infty}$.

For each $n \in \omega$, define $S_n = p^n G[p^{n+1}]$ and note that S_{n+1} is a subgroup of the group S_n, and therefore, the quotient group S_n/S_{n+1} is well-defined. We need to consider two cases.

Case 1. There exists a sequence $0 < m_1 < m_2 < \ldots < m_k < \ldots$ of natural numbers such that $|S_{m_k-1}/S_{m_k}| \geq \omega_1$ for every $k \in \omega$. In this case for every $k \in \omega$ one can find a subgroup V_k of size ω_1 of S_{m_k-1} with $V_k \cap S_{m_k} = \{0\}$. Then the family $\{V_k : k \in \omega\}$ is independent by Lemma 3.14. Since V_k is a subgroup of S_{m_k-1}, by Lemma 3.13 one can find a subgroup N_k of G such that $N_k \cong K_{p^{m_k}}$ and $N_k[p] = V_k$. Now by Lemma 3.12 the family $\{N_k : k \in \omega\}$ is independent. The subgroup $\bigoplus_k N_k$ of G generated by the family $\{N_k : k \in \omega\}$ obviously contains a copy of $K_{p^\infty} = \bigoplus_{k \in \omega} K_{p^k}$.

Case 2. There exists $m_0 \in \omega$ such that $|S_m/S_{m+1}| \leq \omega$ for all $m \geq m_0$. Note that in this case $|S_{m_0}/S_{m_0+k}| \leq \omega$ for every $k \in \omega$. Since $p^{m_0+1} \in \mathfrak{R}(G)$, Lemma 5.6 yields that $G[p^{m_0+1}]$ contains an isomorphic copy of $K_{p^{m_0+1}}$, and thus $S_{m_0} = p^{m_0}G[p^{m_0+1}]$ contains an isomorphic copy of K_p. Therefore, there exists an infinite independent family $\{V_k : k \in \omega\}$ in S_{m_0} with $V_k \cong K_p$ for all k. For every $k \in \omega$ consider the subgroup $W_k = V_k \cap S_{m_0+k}$ of V_k. The quotient group V_k/W_k is naturally isomorphic to a subgroup of S_{m_0}/S_{m_0+k}, and thus $|V_k/W_k| \leq |S_{m_0}/S_{m_0+k}| \leq \omega$ for every $k \in \omega$. Since $V_k \cong K_p$, this yields $W_k \cong K_p$ for every $k \in \omega$. Since W_k is also a subgroup

of S_{m_0+k}, by Lemma 3.13 we get a subgroup N_k of G such that $N_k \cong K_{p^{m_0+k+1}}$ and $N_k[p] = W_k$. Since the family $\{V_k : k \in \omega\}$ was chosen independent, the family $\{W_k : k \in \omega\}$ is independent too. Hence Lemma 3.12 ensures that also the family $\{N_k : k \in \omega\}$ is independent. Therefore it generates a subgroup of G isomorphic to $\bigoplus_{k<\omega} N_k \cong \bigoplus_{k\in\omega} K_{p^{m_0+k+1}}$. It remains only to note that the latter group contains an isomorphic copy of the group K_{p^∞}. □

Lemma 5.8. *Let G be an infinite Abelian group satisfying conditions* **PS** *and* **CC**. *Then G contains a subgroup algebraically isomorphic to $K(G)$.*

Proof. By Lemma 5.5, G contains an isomorphic copy L_0 of $K_0(G)$. By Lemma 5.7, G contains an isomorphic copy L_p of $K_p(G)$ for each $p \in \mathbf{P}$. It remains only to notice that the family $\{L_p : p \in \mathbf{P}\} \cup \{L_0\}$ of subgroups of G is independent, and therefore G contains

$$\bigoplus\{L_p : p \in \mathbf{P}\} \oplus L_0 \cong \bigoplus\{K_p(G) : p \in \mathbf{P}\} \oplus K_0(G) = K(G).$$

□

Lemma 5.9. *Let $\kappa \geq \omega_2$ be a cardinal, G an infinite Abelian group of size at most κ and $H_\kappa = (\mathbb{Q}/\mathbb{Z})^{(\kappa)} \oplus \mathbb{Q}^{(\kappa)}$.*

(i) If G is a non-torsion group that satisfies **PS**, *then for every monomorphism $\phi : K_0 \to H_\kappa$ there exists a monomorphism $\varphi : G \to H_\kappa$ such that $\phi(K_0) \subseteq \varphi(G)$.*

(ii) If G satisfies both **PS** *and* **CC**, *then for every monomorphism $\phi : K(G) \to H_\kappa$ there exists a monomorphism $\varphi : G \to H_\kappa$ such that $\phi(K(G)) \subseteq \varphi(G)$.*

Proof. (i) Assume that G is not torsion and satisfies **PS**. Then $r(G) \geq \mathfrak{c}$, and so G contains a subgroup algebraically isomorphic to K_0. Thus we shall assume, without loss of generality, that K_0 is a subgroup of G. Hence it suffices to find a monomorphism $\varphi : G \to H_\kappa$ extending ϕ. Since $|G| \leq \kappa = r(H_\kappa)$, $r(K_0) = \omega_1 < \omega_2 \leq \kappa = r(H_\kappa)$, $|G| \leq \kappa = r_p(H_\kappa)$ and $r_p(K_0) = 0 \leq \kappa = r_p(H_\kappa)$ for every $p \in \mathbf{P}$, the conclusion of item (i) of our lemma follows from Lemma 3.17 (applied to $G' = K_0$ and $H = H_\kappa$).

(ii) Due to Lemma 5.8 we can assume, without loss of generality, that $K(G)$ is a subgroup of G. Hence it suffices to find a monomorphism $\varphi : G \to H_\kappa$ extending ϕ. Since $|G| \leq \kappa = r(H_\kappa)$, $r(K(G)) \leq \omega_1 < \omega_2 \leq \kappa = r(H_\kappa)$, $|G| \leq \kappa = r_p(H_\kappa)$ and $r_p(K(G)) \leq \omega_1 < \omega_2 \leq \kappa = r_p(H_\kappa)$ for every $p \in \mathbf{P}$, the conclusion of item (ii) of our lemma follows from Lemma 3.17 (applied to $G' = K(G)$ and $H = H_\kappa$). □

6. Forcing

Let $\kappa \geq \omega_2$ be a fixed cardinal. For typographical reasons let H_κ denote the group $\mathbb{Q}^{(\kappa)} \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)}$. Note that our definition of the group H_κ is absolute, i.e. its meaning does not change depending on the model. Our forcing construction uses some ideas from Malyhin's exposition [38, 5.4] of forcing notion due to Hajnal and Juhász [26].

Definition 6.1. (1) Let $\mathbb{P}_\kappa$ be the set of all structures $p = \langle \alpha^p, H^p, \pi^p, \mathcal{E}^p \rangle$ where:
(i_p) $\alpha^p \in \omega_1$,
(ii_p) H^p is a countable subgroup of H_κ,
(iii_p) $\pi^p : H^p \to \mathbb{T}^{\alpha^p}$ is a group homomorphism,
(iv_p) $\mathcal{E}^p \in [[H^p]^\omega]^{\leq\omega}$, and
(v_p) if $A \in [\alpha^p]^{<\omega}$, $\phi \in \mathbb{T}[0]^A$, $k \in \omega \setminus \{0\}$, $E \in \mathcal{E}^p$ and the set

$$E_{A,\phi,k,\pi^p} = \{h \in E : \forall \alpha \in A \ |\pi^p(h)(\alpha) - \phi(\alpha)| < 1/k\}$$

is infinite, then $E_{A,\phi,k,\pi^p} \in \mathcal{E}^p$.

(2) For $p, q \in \mathbb{P}_\kappa$ we define $q \leq p$ provided that the following holds:
(i_p^q) $\alpha^p \leq \alpha^q$,
(ii_p^q) $H^p \subseteq H^q$,
(iii_p^q) $\pi^q(h) \restriction_{\alpha_p} = \pi^p(h)$ for every $h \in H^p$, and
(iv_p^q) if $n \in \omega$, $A \in [\alpha^q \setminus \alpha^p]^{<\omega}$, $\phi \in \mathbb{T}[n]^A$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}^p$ is almost n-torsion in H_κ, then the set E_{A,ϕ,k,π^q} is infinite.

We should note explicitly that we allow $A = \emptyset$ and $\phi = \emptyset$ in the definition of the set E_{A,ϕ,k,π^p} in condition (v_p), and we define $E_{\emptyset,\emptyset,k,\pi^p} = E$. (Incidentally, the last equality also follows from the formal definition of $E_{\emptyset,\emptyset,k,\pi^p}$ since the defining restriction simply vanishes.) Observe that condition (iv_p^q) is vacuously satisfied when $\alpha^q = \alpha^p$. Furthermore, if $\alpha^q = \alpha^p$, then condition (iii_p^q) simply means that the homomorphism $\pi^q : H^q \to \mathbb{T}^{\alpha^q} = \mathbb{T}^{\alpha^p}$ is an extension of the homomorphism $\pi^p : H^p \to \mathbb{T}^{\alpha^p}$ over $H^q \supseteq H^p$.

Lemma 6.2. *$(\mathbb{P}_\kappa, \leq)$ is a partially ordered set.*

Proof. It is clear that the relation $\leq$ is reflexive and asymmetric, so it remains only to check transitivity of $\leq$.

Let $p, q, r \in \mathbb{P}_\kappa$, $p \leq q$ and $q \leq r$. Conditions (i_p^r), (ii_p^r) and (iii_p^r) immediately follow from correspondent pairs of conditions (i_p^q), (i_q^r), (ii_p^q), (ii_q^r), (iii_p^q) and (iii_q^r).

Let us check (iv_p^r). Assume $E \in \mathcal{E}^p$ is almost n-torsion in H_κ for some $n \in \omega$, $A \in [\alpha^r \setminus \alpha^p]^{<\omega}$, $\phi \in \mathbb{T}[n]^A$ and $k \in \omega \setminus \{0\}$. We need to show that the set E_{A,ϕ,k,π^r} is infinite. Define $A^q = A \cap (\alpha^q \setminus \alpha^p)$, $A^r = A \cap (\alpha^r \setminus \alpha^q)$, $\phi^q = \phi \restriction_{A^q}$ and $\phi^r = \phi \restriction_{A^r}$. (Note that we cannot exclude the case $A^q = \phi^q = \emptyset$ or $A^r = \phi^r = \emptyset$ or even both.) Apply condition (iv_p^q) to $A^q \in [\alpha^q \setminus \alpha^p]^{<\omega}$, $\phi^q \in \mathbb{T}[n]^{A^q}$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}^p$ to conclude that the set E_{A^q,ϕ^q,k,π^q} is infinite, and condition (v_q) now implies that $E_{A^q,\phi^q,k,\pi^q} \in \mathcal{E}^q$. Note that E_{A^q,ϕ^q,k,π^q} is an infinite subset of an almost n-torsion set E, so E_{A^q,ϕ^q,k,π^q} itself is almost n-torsion. We can now apply condition (iv_q^r) to $A^r \in [\alpha^r \setminus \alpha^q]^{<\omega}$, $\phi^r \in \mathbb{T}[n]^{A^r}$, $k \in \omega \setminus \{0\}$ and $E_{A^q,\phi^q,k,\pi^q} \in \mathcal{E}^q$ to conclude that the set $(E^q_{A^q,\phi^q,k,\pi^q})_{A^r,\phi^r,k,\pi^r}$ is infinite. Finally, note that $(E_{A^q,\phi^q,k,\pi^q})_{A^r,\phi^r,k,\pi^r} = E_{A,\phi,k,\pi^r}$. □

Lemma 6.3. *If $\{p_n : n \in \omega\} \subseteq \mathbb{P}_\kappa$ and $p_0 \geq p_1 \geq \cdots \geq p_n \geq p_{n+1} \geq \cdots$, then there exists $p \in \mathbb{P}_\kappa$ such that $\alpha^p = \bigcup\{\alpha^{p_n} : n \in \omega\}$ and $p \leq p_n$ for all $n \in \omega$.*

Proof. Let $\alpha^p = \bigcup\{\alpha^{p_n} : n \in \omega\}$, $H^p = \bigcup\{H^{p_n} : n \in \omega\}$, $\mathcal{E}^p = \bigcup\{\mathcal{E}^{p_n} : n \in \omega\}$ and $\pi^p : H^p \to \mathbb{T}^{\alpha^p}$ be the map defined by $\pi^p(h) = \bigcup\{\pi^{p_n}(h) : n \in \omega, h \in H^{p_n}\}$ for each $h \in H^p$. Conditions (i_p) through (iv_p) are straightforward. To check (v_p), assume that $A \in [\alpha^p]^{<\omega}$, $\phi \in \mathbb{T}[0]^A$, $k \in \omega \setminus \{0\}$, $E \in \mathcal{E}^p$ and the set E_{A,ϕ,k,π^p} is infinite. Since $A \subseteq \alpha^p = \bigcup\{\alpha^{p_n} : n \in \omega\}$ is a finite set and $E \in \mathcal{E}^p = \bigcup\{\mathcal{E}^{p_n} : n \in \omega\}$, there exists $n \in \omega$ such that $A \subseteq \alpha^{p_n}$ and $E \in \mathcal{E}^{p_n}$. Observe that $E_{A,\phi,k,\pi^p} = E_{A,\phi,k,\pi^{p_n}}$ and thus the latter set is infinite. Now (v_{p_n}) implies $E_{A,\phi,k,\pi^p} = E_{A,\phi,k,\pi^{p_n}} \in \mathcal{E}^{p_n} \subseteq \mathcal{E}^p$. Thus, $p = \langle \alpha^p, H^p, \pi^p, \mathcal{E}^p\rangle \in \mathbb{P}_\kappa$.

Let us check that $p \leq p_n$ for all $n \in \omega$. Fix $n \in \omega$. Conditions $(\mathrm{i}^p_{p_n})$, $(\mathrm{ii}^p_{p_n})$ and $(\mathrm{iii}^p_{p_n})$ are clear. To check $(\mathrm{iv}^p_{p_n})$, assume that $i \in \omega$, $A \in [\alpha^p \setminus \alpha^{p_n}]^{<\omega}$, $\phi \in \mathbb{T}[n]^A$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}^{p_n}$ is almost i-torsion in H_κ. We need to show that E_{A,ϕ,k,π^p} is an infinite set. Pick $m \in \omega$ such that $i < m$ and $A \in [\alpha^{p_m} \setminus \alpha^{p_n}]^{<\omega}$. Condition $(\mathrm{iv}^{p_m}_{p_n})$ implies that the set $E_{A,\phi,k,\pi^{p_m}}$ is infinite. Since $E_{A,\phi,k,\pi^{p_m}} = E_{A,\phi,k,\pi^p}$, the result follows. □

Lemma 6.4. *Assume CH. Then $(\mathbb{P}_\kappa, \leq)$ is ω_2-c.c.*

Proof. Suppose that $\{p_\beta : \beta \in \omega_2\} \subseteq \mathbb{P}_\kappa$. There exist $\Gamma \in [\omega_2]^{\omega_2}$ and $\alpha \in \omega_1$ such that $\alpha^{p_\gamma} = \alpha$ for all $\gamma \in \Gamma$. Since CH holds, by the Δ-system lemma applied to the family $\{H^{p_\gamma} : \gamma \in \Gamma\}$ there exist $\Gamma' \in [\Gamma]^{\omega_2}$ and $K \subseteq H_\kappa$ such that $H^{p_\beta} \cap H^{p_\gamma} = K$ whenever $\beta, \gamma \in \Gamma'$ and $\beta \neq \gamma$ [37, Chapter II, Theorem 1.6]. Applying CH once again, we can find $\Gamma'' \in [\Gamma']^{\omega_2}$ such that $\pi^{p_\beta}(h) = \pi^{p_\gamma}(h)$ whenever $h \in K$, $\beta, \gamma \in \Gamma''$ and $\beta \neq \gamma$. We now claim that the family $\{p_\gamma : \gamma \in \Gamma''\}$ consists of pairwise compatible conditions. Indeed, let $\beta, \gamma \in \Gamma''$ and $\beta \neq \gamma$. Define $\alpha^r = \alpha$, $H^r = H^{p_\beta} + H^{p_\gamma}$, $\mathcal{E}^r = \mathcal{E}^{p_\beta} \cup \mathcal{E}^{p_\gamma}$ and let $\pi^r : H^r \to \mathbb{T}^{\alpha^r} = \mathbb{T}^{\alpha}$ be the homomorphism extending both π^{p_β} and π^{p_γ} (Lemma 3.1). It is easy to check that $r = \langle \alpha^r, H^r, \pi^r, \mathcal{E}^r\rangle \in \mathbb{P}_\kappa$, $r \leq p_\beta$ and $r \leq p_\gamma$. □

Lemma 6.5. *If $p \in \mathbb{P}_\kappa$ and $E \in [H_\kappa]^{\leq\omega}$, then there exists $q \in \mathbb{P}_\kappa$ such that $q \leq p$ and $E \subseteq H^q$.*

Proof. If $E \subseteq H^p$, then $q = p$ works. Otherwise define $\alpha^q = \alpha^p$, $H^q = H^p + \langle\!\langle E \rangle\!\rangle$, $\mathcal{E}^q = \mathcal{E}^p$ and let $\pi^q : H^q \to \mathbb{T}^{\alpha^q} = \mathbb{T}^{\alpha^p}$ be any homomorphism extending π^p (such a homomorphism exists by Lemma 3.2 since the group $\mathbb{T}^{\alpha^q}$ is divisible). Now $q = \langle \alpha^q, H^q, \pi^q, \mathcal{E}^q\rangle$ does the job. □

Lemma 6.6. *Given $\alpha^q \in \omega_1$, a countable subgroup H^q of H_κ, a homomorphism $\pi^q : H^q \to \mathbb{T}^{\alpha^q}$ and $\mathcal{E} \in [[H^q]^\omega]^{\leq\omega}$, there exists $\mathcal{E}^q \in [[H^q]^\omega]^{\leq\omega}$ such that $\mathcal{E} \subseteq \mathcal{E}^q$ and condition (v_q) holds.*

Proof. For every family $\mathcal{C} \in [[H^q]^\omega]^{\leq\omega}$ define $\mathcal{C}' = \mathcal{C} \cup \{E_{A,\phi,k,\pi^q} : C \in \mathcal{C}, A \in [\alpha^q]^{<\omega}, \phi \in \mathbb{T}[0]^A, k \in \omega \setminus \{0\}, E_{A,\phi,k,\pi^q} \text{ is infinite}\}$. Clearly $\mathcal{C}' \in [[H^q]^\omega]^{\leq\omega}$. By induction on $n \in \omega$, define $\mathcal{E}_n \in [[H^q]^\omega]^{\leq\omega}$ as follows. Let $\mathcal{E}_0 = \mathcal{E}$, and $\mathcal{E}_{n+1} = \mathcal{E}'_n$ for every $n \in \omega$. It is easy to see that $\mathcal{E}^q = \bigcup\{\mathcal{E}_n : n \in \omega\}$ satisfies the conclusion of our lemma. □

Lemma 6.7. *If $p \in \mathbb{P}_\kappa$ and $E \in [H_\kappa]^\omega$, then there exists $q \in \mathbb{P}_\kappa$ such that $q \leq p$ and $E \in \mathcal{E}^q$.*

Proof. Lemma 6.5 allows us to find $r \in \mathbb{P}_\kappa$ such that $r \leq p$ and $E \subseteq H^r$. In particular, $E \subseteq [H^r]^\omega$. Apply Lemma 6.6 to $\alpha^q = \alpha^r$, $H^q = H^r$, $\pi^q = \pi^r$ and $\mathcal{E} = \mathcal{E}^r \cup \{E\}$ to get $\mathcal{E}^q \in [[H^q]^\omega]^{\leq\omega}$ such that $\mathcal{E} \subseteq \mathcal{E}^q$ and condition (v_q) holds. Now $q = \langle \alpha^q, H^q, \pi^q, \mathcal{E}^q \rangle \in \mathbb{P}_\kappa$ is as required. □

Lemma 6.8. *If $p \in \mathbb{P}_\kappa$, $g \in H^p$ and $g \neq 0$, then there exists $q \in \mathbb{P}_\kappa$ such that $q \leq p$, $\alpha^q = \alpha^p + 1$ and $\pi^q(g) \neq 0$.*

Proof. Let $\alpha^q = \alpha^p + 1$ and $H^q = H^p$. Apply Lemma 4.9 to $G = H^p$, g and $\mathcal{E} = \mathcal{E}^p$ to get a homomorphism $\varpi : G \to \mathbb{T}$ as in the conclusion of this lemma. Define now a homomorphism $\pi^q : G \to \mathbb{T}^{\alpha^q} = \mathbb{T}^{\alpha^p+1} = \mathbb{T}^{\alpha^p} \times \mathbb{T}$ by $\pi^q(h) = (\pi^p(h), \varpi(h))$ for $h \in G$. Since $\varpi(g) \neq 0$, we have $\pi^q(g) \neq 0$, as required. Finally, apply Lemma 6.6 to $\mathcal{E} = \mathcal{E}^p$ to choose $\mathcal{E}^q \in [[H^q]^\omega]^{\leq\omega}$ such that $\mathcal{E}^p \subseteq \mathcal{E}^q$ and condition (v_q) holds. It is clear from our construction that $q = \langle \alpha^q, H^q, \pi^q, \mathcal{E}^q \rangle \in \mathbb{P}_\kappa$.

Let us prove that $q \leq p$. Conditions (i^q_p), (ii^q_p) and (iii^q_p) are immediate from our construction. It remains only to verify condition (iv^q_p). Assume that $n \in \omega$, $A \in [\alpha^q \setminus \alpha^p]^{<\omega}$, $\phi \in \mathbb{T}[n]^A$, $k \in \omega \setminus \{0\}$ and $E \in \mathcal{E}^p$ is almost n-torsion in H_κ. Since $\alpha^q \setminus \alpha^p = \{\alpha^p\}$, either $A = \emptyset$ or $A = \{\alpha^p\}$. If $A = \emptyset$, then $\phi = \emptyset$, $E_{A,\phi,k,\pi^q} = E_{\emptyset,\emptyset,k,\pi^q} = E$ and the latter set is infinite. Otherwise $A = \{\alpha^p\}$ and $\phi = \{\langle \alpha^p, z \rangle\}$ for some $z \in \mathbb{T}[n]$. By our choice of ϖ the set $E_{A,\phi,k,\pi^q} = \{h \in E : |\pi^q(h)(\alpha^p) - \phi(\alpha^p)| < 1/k\} = \{h \in E : |\varpi(h) - z| < 1/k\}$ is infinite. □

Lemma 6.9. *For every $p \in \mathbb{P}_\kappa$ there exists $q \in \mathbb{P}_\kappa$ such that $q \leq p$ and $\pi^q \restriction_{H^p} : H^p \to \mathbb{T}^{\alpha^q}$ is a monomorphism.*

Proof. Without loss of generality, we may assume that H^p is infinite (Lemma 6.5). Let $H^p = \{h_n : n \in \omega\}$ be an enumeration of points of H^p. By induction on n, we repeatedly use Lemma 6.8 to obtain a decreasing sequence $p = p_0 \geq p_1 \geq \cdots \geq p_n \geq p_{n+1} \geq \ldots$ of elements of $\mathbb{P}_\kappa$ such that $\pi^{p_n}(h_n) \neq 0$. Choose $p \in \mathbb{P}_\kappa$ such that $p \leq p_n$ for all n (Lemma 6.3). Condition $(\mathrm{iii}^p_{p_n})$ implies that $\pi^p(h_n) \neq 0$ for each n. Thus the kernel of $\pi^q \restriction_{H^p} : H^p \to \mathbb{T}^{\alpha^q}$ is trivial. □

Lemma 6.10. *If $p \in \mathbb{P}_\kappa$ and $\beta \in \omega_1$, then there exists $q \in \mathbb{P}_\kappa$ such that $q \leq p$ and $\beta \in \alpha^q$.*

Proof. Lemma 6.5 guarantees the existence of $p_{\alpha^p} \in \mathbb{P}_\kappa$ such that $p_{\alpha^p} \leq p$ and $H^{p_{\alpha^p}} \neq \{0\}$. Pick $g \in H^{p_{\alpha^p}}$ with $g \neq 0$. By transfinite recursion on $\gamma \in \omega_1 \setminus \alpha^p$, we will construct $p_\gamma \in \mathbb{P}_\kappa$ such that:

(1_γ) $\alpha^{p_\gamma} = \gamma$,

(2_γ) $\alpha < \alpha' \leq \gamma$ implies $p_{\alpha'} \leq p_\alpha$.

Clearly $p_{\alpha^p} \in \mathbb{P}_\kappa$ satisfies (1_{α^p}) and (2_{α^p}). Suppose now that we have already constructed $\{p_\alpha : \alpha \in \gamma \setminus \alpha^p\}$ for some $\gamma \in \omega_1 \setminus \alpha^p$.

If γ is a limit ordinal, then we can pick a strictly increasing sequence $\{\gamma_n : n \in \omega\}$ of ordinals cofinal in γ and apply Lemma 6.3 to find $p_\gamma \in \mathbb{P}_\kappa$ such that $\alpha^{p_\gamma} = \bigcup\{\alpha^{p_{\gamma_n}} : n \in \omega\} = \bigcup\{\gamma_n : n \in \omega\} = \gamma$ and $p_\gamma \leq p_{\gamma_n}$ for all $n \in \omega$. If $\delta < \gamma$, then $\delta < \gamma_n$ for some n, and thus $p_\gamma \leq p_{\gamma_n} \leq p_\delta$. This yields (2_γ).

If $\gamma = \delta + 1$ is a successor ordinal, then $p_\delta \leq p_{\alpha^p}$ implies $g \in H^{p_{\alpha^p}} \subseteq H^{p_\delta}$, and therefore we can apply Lemma 6.8 with p_δ as p (and our g) to find $p_\gamma \in \mathbb{P}_\kappa$ such that $\alpha^{p_\gamma} = \alpha^{p_\delta} + 1 = \delta + 1 = \gamma$ and $p_\gamma \leq p_\delta$.

To finish the proof, notice that $p_{\beta+1}$ can be taken as q. □

Recall that $K_n = \mathbb{T}[n]^{(\omega_1)}$ when $n > 1$ and $K_0 = \mathbb{Z}^{(\omega_1)}$.

Lemma 6.11. *Suppose that $p \in \mathbb{P}_\kappa$, $\beta \in \omega_1$, $n \in \omega \setminus \{1\}$, $y \in \mathbb{T}^\beta$ with $ny = 0$, and K is an uncountable subgroup of H_κ isomorphic to K_n. Then there exist $q \in \mathbb{P}_\kappa$ and $x \in K \cap H^q$ such that $\beta \in \alpha^q$ and $\pi^q(x) \restriction_\beta = y$.*

Proof. By our assumption, we can write $K = \bigoplus_{\alpha\in\omega_1} C_\alpha$, where each C_α is algebraically isomorphic to $\mathbb{T}[n]$ when $n > 1$ and to $\mathbb{Z}$ when $n = 0$. By Lemma 6.10, there exists $r \in \mathbb{P}_\kappa$ such that $r \leq p$ and $\beta \in \alpha^r$. Pick arbitrarily element $y' \in \mathbb{T}^{\alpha^r}$ with $y' \restriction_\beta = y$ and $ny' = 0$. Observe that there must exist $\alpha \in \omega_1$ such that $C_\alpha \cap H^r = \{0\}$. Indeed, otherwise, for each $\alpha \in \omega_1$ there would exist $h_\alpha \in C_\alpha \cap H^p \setminus \{0\}$, and all these elements h_α must be pairwise distinct (because $C_\alpha \cap C_\beta = \{0\}$ for $\alpha \neq \beta$), thereby implying $|H^r| = \omega_1$, a contradiction. Fix $\alpha \in \omega_1$ with $C_\alpha \cap H^r = \{0\}$, and let x be any generator of the cyclic group C_α. By the choice of x, there exists a group homomorphism $\varphi : \langle\!\langle x \rangle\!\rangle \to \langle\!\langle y' \rangle\!\rangle$ such that $\varphi(x) = y'$. Define $\alpha^q = \alpha^r$, $\mathcal{E}^q = \mathcal{E}^r$, $H^q = H^r + \langle\!\langle x \rangle\!\rangle$, and let $\pi^q : H^r \to \mathbb{T}^{\alpha^q} = \mathbb{T}^{\alpha^r}$ be a group homomorphism which extends both π^r and φ. (Such a homomorphism exists by Lemma 3.1 because $\langle\!\langle x \rangle\!\rangle \cap H^r = C_\alpha \cap H^r = \{0\}$.) Then $q = \langle \alpha^q, H^q, \pi^q, \mathcal{E}^q \rangle$ is as required. □

7. Defining a monomorphism $\pi : H_\kappa \to \mathbb{T}^{\omega_1}$ in the generic extension $M_\kappa[\mathbb{G}]$

Let M_κ be a model of ZFC such that $\kappa \in M_\kappa$, $\mathbb{P}_\kappa \in M_\kappa$ and both $\mathfrak{c} = \omega_1$ and $2^{\omega_1} = \kappa$ hold in M_κ. Let $\mathbb{G} \subseteq \mathbb{P}_\kappa$ be a set $\mathbb{P}_\kappa$-generic over M_κ, and $M_\kappa[\mathbb{G}]$ the generic extension of M_κ via $\mathbb{G}$.

Lemma 7.1. *Forcing with $\mathbb{P}_\kappa$ preserves cardinals.*

Proof. From Lemma 6.3 it follows that the poset $\mathbb{P}_\kappa$ is countably closed (or ω_1-closed in the terminology of [37]), and thus forcing with $\mathbb{P}_\kappa$ preserves cardinal ω_1 [37, Chapter VII, Corollary 6.15]. From Lemma 6.4, the fact that CH holds in the ground model M_κ and [37, Chapter VII, Lemma 6.9] one concludes that forcing with $\mathbb{P}_\kappa$ does not collapse cardinals greater or equal than ω_2. It now follows that all cardinals are preserved by $\mathbb{P}_\kappa$. □

Lemma 7.2. *$\mathbb{P}_\kappa$ does not introduce new countable sets. That is, if $B \in M_\kappa$, $C \in M_\kappa[\mathbb{G}]$, $C \subseteq B$ and C is countable in $M_\kappa[\mathbb{G}]$, then $C \in M_\kappa$.*

Proof. Since $C \subseteq B$ and C is countable in $M_\kappa[\mathbb{G}]$, there exists a function $f : \omega \to B$ such that $C = f(\omega)$ and $f \in M_\kappa[\mathbb{G}]$. Lemma 6.3 says that $\mathbb{P}_\kappa$ is countably closed (or ω_1-closed in the terminology of [37]), so applying [37, Chapter VII, Theorem 6.14] (with $A = \omega$ and $\lambda = \omega_1$) yields $f \in M_\kappa$. Since $\omega \in M_\kappa$, $f \in M_\kappa$ and M_κ is a model of ZFC, it now follows that $C = f(\omega) \in M_\kappa$. □

From Lemmas 6.5 and Lemma 6.10 we obtain

Lemma 7.3. *In $M_\kappa[\mathbb{G}]$, we have $H_\kappa = \bigcup\{H^p : p \in \mathbb{G}\}$ and $\omega_1 = \bigcup\{\alpha^p : p \in \mathbb{G}\}$.*

In $M_\kappa[\mathbb{G}]$, for each $h \in H_\kappa$ define $\pi(h) = \bigcup\{\pi^p(h) : p \in \mathbb{G}, h \in H^p\}$.

Lemma 7.4. *(i) $\pi(h) \in \mathbb{T}^{\omega_1}$ for every $h \in H_\kappa$.*
(ii) If $p \in \mathbb{G}$ and $h \in H^p$, then $\pi(h) \restriction_{\alpha^p} = \pi^p(h)$.

Proof. This follows via standard argument from (iii$_p$), (iii^q_p) and the fact that $\mathbb{G}$ consists of pairwise compatible elements. □

Lemma 7.5. *In $M_\kappa[\mathbb{G}]$, $\pi : H_\kappa \to \mathbb{T}^{\omega_1}$ is a monomorphism.*

Proof. Conditions (iii$_p$) and (iii^q_p) imply that π is a group homomorphism. Lemma 6.8 and standard density argument yields that the kernel of π is trivial. □

By the above lemma, $\pi(H_\kappa)$ is a subgroup of $\mathbb{T}^{\omega_1}$ algebraically isomorphic to H_κ. In $M_\kappa[\mathbb{G}]$, we will always consider $\mathbb{T}^{\omega_1}$ equipped with the Tychonoff product topology. Recall that $\mathbb{T}^{\omega_1}$ is compact.

Lemma 7.6. *In $M_\kappa[\mathbb{G}]$, $\mathfrak{c} = \omega_1$ and $2^{\omega_1} = 2^{\mathfrak{c}} = \kappa$.*

Proof. According to Lemma 7.2, the values of $2^\omega = \mathfrak{c}$ in both models M_κ and $M_\kappa[\mathbb{G}]$ coincide. Since $\omega_1 = \mathfrak{c}$ holds in M_κ, and ω_1 is preserved (Lemma 7.1), we conclude that $\omega_1 = \mathfrak{c}$ holds in $M_\kappa[\mathbb{G}]$ as well.

Lemma 7.5 implies that $\kappa = |H_\kappa| \leq |\mathbb{T}^{\omega_1}| = \mathfrak{c}^{\omega_1} \leq (2^{\omega_1})^{\omega_1} = 2^{\omega_1}$ holds in $M_\kappa[\mathbb{G}]$. It remains only to show that the reverse inequality $2^{\omega_1} \leq \kappa$ holds in $M_\kappa[\mathbb{G}]$.

First, observe that, in M_κ, one has $\kappa^{\omega_1} = (2^{\omega_1})^{\omega_1} = 2^{\omega_1} = \kappa$. In particular, $\kappa^\omega = \kappa$ holds in M_κ.

Second, we claim that $|\mathbb{P}_\kappa| = \kappa$ in M_κ. Indeed, in M_κ, we have $|H_\kappa| = \kappa$, which implies $|[H_\kappa]^{\leq\omega}| \leq \kappa^\omega = \kappa$ and $|[[H_\kappa]^\omega]^{\leq\omega}| \leq \kappa^\omega = \kappa$, and thus the number of group homomorphisms $\pi : H \to \mathbb{T}^\alpha$ with $H \in [H_\kappa]^{\leq\omega}$ and $\alpha \in \omega_1$ is bounded by $\kappa \cdot \omega_1 \cdot \mathfrak{c}^\omega = \kappa$.

Third, since CH holds in M_κ, every antichain in $(\mathbb{P}_\kappa, \leq)$ that is an element of M_κ has size at most ω_1 (Lemma 6.4). Therefore, in M_κ, the total number of antichains in $(\mathbb{P}_\kappa, \leq)$ does not exceed $|\mathbb{P}_\kappa^{\omega_1}| = \kappa^{\omega_1} = \kappa$. This yields that the number of nice $\mathbb{P}_\kappa$-names (in the sense of [37, Chapter VII, Definition 5.11]) in M_κ for subsets of ω_1 does not exceed $\kappa^{\omega_1} = \kappa$. Arguing as in the end of the proof of [37, Chapter VII, Lemma 5.13], we can now conclude that $2^{\omega_1} \leq \kappa$ holds in $M_\kappa[\mathbb{G}]$. □

8. $\pi(H_\kappa)$ has no non-trivial convergent sequences

Lemma 8.1. *In the generic extension $M_\kappa[\mathbb{G}]$, the following holds: If $n \in \omega \setminus \{1\}$ and E is an almost n-torsion subset of H_κ, then there exists $\gamma \in \omega_1$ such that the set $\{\pi(h)\restriction_{\omega_1\setminus\gamma}: h \in E\}$ is dense in $\mathbb{T}[n]^{\omega_1\setminus\gamma}$.*

Proof. For $n \in \omega$, let E be an almost n-torsion subset of H_κ. Using Lemma 6.7 and a standard density argument we conclude that there exists $p \in \mathbb{G}$ such that $E \in \mathcal{E}^p$. We claim that $\gamma = \alpha_p$ works. Since $\mathbb{T}[0]^{\omega_1\setminus\gamma}$ is dense in $\mathbb{T}^{\omega_1\setminus\gamma}$, it suffices to check the following property: If $A \in [\omega_1 \setminus \gamma]^{<\omega}$, $\phi \in \mathbb{T}[n]^A$ and $k \in \omega \setminus \{0\}$, then there exists $h \in E$ such that $|\pi(h)(\alpha) - \phi(\alpha)| < 1/k$ for each $\alpha \in A$. By Lemma 6.10 and the standard density argument, one can find $q \in \mathbb{G}$ such that $q \leq p$ and $A \subseteq \alpha^q$. Observe that $A \in [\alpha^q \setminus \gamma]^{<\omega} = [\alpha^q \setminus \alpha^p]^{<\omega}$, and the condition (iv^q_p) implies that the set E_{A,ϕ,k,π^q} is infinite. Pick arbitrarily $h \in E_{A,\phi,k,\pi^q}$ and note that, according to the definition of E_{A,ϕ,k,π^q}, one has $h \in E$ and $|\pi^q(h)(\alpha) - \phi(\alpha)| < 1/k$ for each $\alpha \in A$. Finally, $A \subseteq \alpha^q$ and Lemma 7.4(ii) yield $\pi^q(h)(\alpha) = \pi(h)(\alpha)$ for every $\alpha \in A$. □

Lemma 8.2. *In $M_\kappa[\mathbb{G}]$, the group $\pi(H_\kappa)$ does not have non-trivial convergent sequences.*

Proof. Let $S \in M_\kappa[\mathbb{G}]$ be an infinite subset of H_κ. According to Lemma 4.5, there exist a natural number $n \in \omega \setminus \{1\}$, $E \in [H_\kappa]^\omega$ and $h \in H_\kappa$ so that $h+E \subseteq S$ and E is almost n-torsion in H_κ. By Lemma 8.1, the set $E_\gamma = \{\pi(h)\restriction_{\omega_1\setminus\gamma}: h \in E\}$ is dense in $\mathbb{T}[n]^{\omega_1\setminus\gamma}$ for some $\gamma \in \omega_1$. In particular, E_γ cannot be a convergent sequence. Since E_γ is the image of the set $\pi(E)$ under the continuous projection from $\mathbb{T}^{\omega_1}$ to $\mathbb{T}^{\omega_1\setminus\gamma}$, it follows that $\pi(E)$ cannot be a convergent sequence as well. Therefore, $\pi(h)+\pi(E)$ is not a convergent sequence either. Since π is a monomorphism (Lemma 7.5), we have $\pi(S) \supseteq \pi(h + E) = \pi(h) + \pi(E)$, which implies that $\pi(S)$ cannot be a convergent sequence. □

9. Hereditary separability of $\pi(H_\kappa)$

Our goal of proving that $\pi(H_\kappa)$ is hereditarily separable would be trivial if $\pi(H_\kappa)$ satisfied one of the well-known sufficient conditions for hereditary separability like HFD or HFD_w (see survey [33]). Unfortunately, due to algebraic restrictions imposed by almost n-torsion sets (for $n \geq 2$), $\pi(H_\kappa)$ does not have the above mentioned properties. So we need to carry out a more delicate analysis in order to prove hereditary separability of $\pi(H_\kappa)$.

Definition 9.1. Let Z be a topological space.

(i) For each $\gamma \in \omega$, we define the projection map $j^Z_\gamma : Z^{\omega_1} \to Z^{\omega_1\setminus\gamma}$ by $j^Z_\gamma(z) = z\restriction_{\omega_1\setminus\gamma}$ for $z \in Z^{\omega_1}$.

(ii) A subset X of Z^{ω_1} will be called *finally separable in Z^{ω_1}* provided that there exists $\gamma \in \omega_1$ such that $j^Z_\gamma(X)$ is separable (in the subspace topology of $Z^{\omega_1\setminus\gamma}$). In other words, X is finally separable in Z^{ω_1} if there exist $\gamma \in \omega_1$ and $E \in [X]^{\leq\omega}$

such that the set $j^Z_\gamma(E)$ is dense in $j^Z_\gamma(X)$ (considered with the subspace topology of $Z^{\omega_1\setminus\gamma}$).

(iii) A subset Y of Z^{ω_1} will be called *hereditarily finally separable in* Z^{ω_1}, or shortly *HFS in* Z^{ω_1}, provided that each subset X of Y is finally separable in Z^{ω_1}.

Lemma 9.2. *Let Z is a space and X a subspace of Z^{ω_1}. Suppose that there exist $\gamma\in\omega_1$ and $E\in[X]^{\le\omega}$ such that $j^Z_\gamma(E)$ is dense in $Z^{\omega_1\setminus\gamma}$. Then X is finally separable in Z^{ω_1}.*

Proof. Indeed, $j^Z_\gamma(E)$ is a countable dense subset of $j^Z_\gamma(X)$. □

Our next lemma demonstrates that these notions are quite appropriate:

Lemma 9.3. *Let Z be a separable metric space and Y be a subspace of Z^{ω_1}. Then the following conditions are equivalent:*

(i) Y is hereditarily separable,

(ii) Y is HFS in Z^{ω_1}.

Proof. [8] Clearly (i) implies (ii) since hereditary separability is preserved by continuous maps, and each projection j^Z_γ is continuous.

Let us now prove that (ii) implies (i). If Y is HFS in Z^{ω_1} and X is a subset of Y, then X itself is HFS in Z^{ω_1}. Therefore, to prove implication (ii) $\to$ (i), it suffices only to check that every set Y that is HFS in Z^{ω_1} is separable (in the subspace topology of Z^{ω_1}).

Fix a countable base $\mathcal{B}$ of Z. For $A\in[\omega_1]^{<\omega}$ and $B\in\mathcal{B}^A$, define $W_{A,B}=\{z\in Z^{\omega_1}: z(\alpha)\in B(\alpha)\text{ for all }\alpha\in A\}$ and $Y_{A,B}=Y\cap W(A,B)$, and also fix $\gamma_{A,B}\in\omega_1$ and $D_{A,B}\in[Y_{A,B}]^{\le\omega}$ such that $j^Z_{\gamma_{A,B}}(D_{A,B})$ is dense in $j^Z_{\gamma_{A,B}}(Y_{A,B})$. This is possible because the set $Y_{A,B}$ is finally dense in Z^{ω_1}, being a subset of the set Y, which in turn is HFS in Z^{ω_1}. Define a function $\sigma:\omega_1\to\omega_1$ by $\sigma(\gamma)=\gamma+1+\sup\{\gamma_{A,B}: A\in[\gamma]^{<\omega}, B\in\mathcal{B}^A\}$ for $\gamma\in\omega_1$. Pick $\gamma_0\in\omega_1$ arbitrarily, and define a sequence $\{\gamma_n: n\in\omega\}$ of countable ordinals via $\gamma_{n+1}=\sigma(\gamma_n)$. Let $\lambda=\sup\{\gamma_n: n\in\omega\}$, and note that $\lambda\in\omega_1$ and $\gamma_1<\gamma_2<\dots<\gamma_n<\dots<\lambda$.

Clearly, $D=\bigcup\{D_{A,B}: A\in[\lambda]^{<\omega}, B\in\mathcal{B}^A\}$ is a countable subset of Y. It remains only to prove that D is dense in Y. Let O be an arbitrarily open subset of Z^{ω_1} with $O\cap Y\ne\emptyset$. We need to show that $O\cap D\ne\emptyset$. There exist $A\in[\omega_1]^{<\omega}$, $B\in\mathcal{B}^A$ and $x\in Y$ such that $x\in W_{A,B}\subseteq O$. Define $A'=A\cap\lambda$, $A''=A\setminus\lambda$, $B'=B\restriction_{A'}$, $B''=B\restriction_{A''}$ and $\mu=\gamma_{A',B'}$. Then $\max(A')<\gamma_n$ for some $n\in\omega$, and hence $\mu=\gamma_{A',B'}\le\gamma_{n+1}<\lambda$ by our definition of λ and γ_n's. Since $A'\in[\lambda]^{<\omega}$, from our definitions of D and B' it follows that $D_{A',B'}\subseteq D$.

Since $W_{A'',B''}$ is a basic open set in Z^{ω_1} and j^Z_μ is an open map, the set $j^Z_\mu(W_{A'',B''})$ is open in $Z^{\omega_1\setminus\mu}$. From $x\in W_{A,B}\subseteq W_{A'',B''}$, one gets $j^Z_\mu(x)\in j^Z_\mu(W_{A'',B''})$. Since $x\in Y\cap W_{A,B}\subseteq Y\cap W_{A',B'}=Y_{A',B'}$, we have $j^Z_\mu(x)\in j^Z_\mu(Y_{A',B'})$, and so $j^Z_\mu(x)\in$

[8] In case Z is the two-point discrete space $\{0,1\}$, this lemma is essentially proved in [33].

$j^Z_\mu(Y_{A',B'}) \cap j^Z_\mu(W_{A'',B''}) \neq \emptyset$. Since $W_{A'',B''}$ is a basic open set in Z^{ω_1} and j^Z_μ is an open map, the set $j^Z_\mu(W_{A'',B''})$ is open in $Z^{\omega_1\setminus\mu}$, which implies that $j^Z_\mu(Y_{A',B'}) \cap j^Z_\mu(W_{A'',B''})$ is a non-empty open subset of $j^Z_\mu(Y_{A',B'})$. By denseness of $j^Z_\mu(D_{A',B'})$ in $j^Z_\mu(Y_{A',B'})$, we conclude that there exists $y \in D_{A',B'}$ such that $j^Z_\mu(y) \in j^Z_\mu(W_{A'',B''})$. Together with $\mu < \lambda \leq \min A''$ this yields $y \in W_{A'',B''}$. Since $y \in D_{A',B'} \subseteq D$ and $y \in D_{A',B'} \subseteq Y_{A',B'} \subseteq W_{A',B'}$, we finally get $y \in D \cap W_{A',B'} \cap W_{A'',B''} = D \cap W_{A,B} \subseteq D \cap O \neq \emptyset$. □

For the reader familiar with the notion of an elementary submodel, we mention in passing that an elegant way to get separability of Y in the proof of the above lemma is to take any countable elementary submodel M of (a sufficiently large fragment of) the universe V containing all relevant information (such as Y, Z etc.), and then observe that the countable set $M \cap Y$ is *automatically* dense in Y.

Lemma 9.4. *Let G be a topological group and $\mathcal{H}$ a countable family of its subgroups such that each $H \in \mathcal{H}$ is hereditarily separable (in the subspace topology). If $X \subseteq G$ is not separable, then there exists $E \in [X]^\omega$ such that $|(g + H) \cap E| \leq 1$ whenever $H \in \mathcal{H}$ and $g \in G$.*

Proof. For a fixed $g \in G$, the translation map that sends $x \in G$ to $g + x$ is a homeomorphism of G onto itself, and so $g + H$ is hereditarily separable for every $H \in \mathcal{H}$. Note that a countable union of hereditarily separable subspaces of G is again hereditarily separable, and thus $Y_C = \bigcup\{g+H : g \in C, H \in \mathcal{H}\}$ is hereditarily separable for every $C \in [H]^{\leq\omega}$. Since our X is not separable, $X \setminus Y_C \neq \emptyset$ for every $C \in [H]^{\leq\omega}$. This allows us, by recursion on $n \in \omega$, to pick $x_n \in X \setminus Y_{\{x_0,\dots,x_{n-1}\}}$. Clearly $E = \{x_n : n \in \omega\}$ has the required properties. □

Lemma 9.5. *Suppose that G is a subgroup of $\mathbb{T}^{\omega_1}$ such that, for all $n > 1$ and each almost n-torsion subset E of G there exists $\gamma \in \omega_1$ such that the set $j^{\mathbb{T}}_\gamma(E)$ is dense in $\mathbb{T}[n]^{\omega_1\setminus\gamma}$.*

Then for every integer $n \geq 2$ the subgroup $G[n] = G \cap \mathbb{T}[n]^{\omega_1}$ of G is hereditarily separable in the subspace topology induced from $\mathbb{T}^{\omega_1}$.

Proof. Let us prove by induction on $n \geq 2$ that $G[n] = G \cap \mathbb{T}[n]^{\omega_1}$ is hereditarily separable.

Basis of induction. Let us prove that $G[2] = G \cap \mathbb{T}[2]^{\omega_1}$ is hereditarily separable. In view of our Lemma 9.3 (with $Z = \mathbb{T}[2]$) it suffices to prove that $G[2] \subseteq \mathbb{T}[2]^{\omega_1}$ is HFS in $\mathbb{T}[2]^{\omega_1}$. Let X be a subset of $G[2]$. If X is finite, then clearly X is finally separable in $\mathbb{T}[2]^{\omega_1}$. Suppose that X is infinite and pick $E \in [X]^\omega$. Since $E \subseteq X \subseteq G[2]$, E is an almost 2-torsion subset of G. By the assumption of our

lemma, there exists $\gamma \in \omega_1$ such that the set $j^{\mathbb{T}}_{\gamma}(E)$ is dense in $\mathbb{T}[2]^{\omega_1 \setminus \gamma}$. Now X is finally separable in $\mathbb{T}[2]^{\omega_1}$ by Lemma 9.2.[9]

Inductive step. Let $n > 2$ and suppose that we have already proved that $G[m]$ is hereditarily separable for all m with $2 \leq m < n$. Let us now prove that $G[n]$ is hereditarily separable. In view of our Lemma 9.3 (with $Z = \mathbb{T}[n]$) it suffices to prove that $G[n] \subseteq \mathbb{T}[n]^{\omega_1}$ is HFS in $\mathbb{T}[n]^{\omega_1}$. Let X be a subset of $G[n]$. If X is separable, then X is finally separable ($\gamma = 0$ works). Suppose that X is not separable. By our inductive hypothesis, $\mathcal{H} = \{G[m] : 2 \leq m < n\}$ is a finite family of hereditarily separable subgroups of G, and so we can apply Lemma 9.4 with this $\mathcal{H}$ and our X to get the set $E \in [X]^{\omega}$ as in the conclusion of Lemma 9.4. Now observe that E is almost n-torsion. By the hypothesis of our lemma, there exists $\gamma \in \omega_1$ such that the set $j^{\mathbb{T}}_{\gamma}(E)$ is dense in $\mathbb{T}[n]^{\omega_1 \setminus \gamma}$, and Lemma 9.2 yields that X is finally separable in $\mathbb{T}[n]^{\omega_1}$. □

Lemma 9.6. *Suppose that G is a subgroup of $\mathbb{T}^{\omega_1}$ such that for all $n \in \omega \setminus \{1\}$ and each almost n-torsion subset E of G there exists $\gamma \in \omega_1$ such that the set $j^{\mathbb{T}}_{\gamma}(E)$ is dense in $\mathbb{T}[n]^{\omega_1 \setminus \gamma}$. Then G is hereditarily separable in the subspace topology induced from $\mathbb{T}^{\omega_1}$.*

Proof. By Lemma 9.5, $G[n]$ is hereditarily separable for every $n \geq 2$. Since a countable union of hereditarily separable subspaces is hereditarily separable, we conclude that the torsion part $t(G) = \bigcup\{G[n] : n \geq 2\}$ of G is hereditarily separable.

According to Lemma 9.3, to prove that G is hereditarily separable it suffices to show that $G \subseteq \mathbb{T}^{\omega_1}$ is HFS in $\mathbb{T}^{\omega_1}$. Let $X \subseteq G$. If X is separable, then it is finally separable ($\gamma = 0$ works). Otherwise we can apply Lemma 9.4 with $\mathcal{H} = \{t(G)\}$ to our X to get $E \in [X]^{\omega}$ as in conclusion of Lemma 9.4. According to Lemma 4.5, there exist an almost n-torsion set $E' \subseteq G$ and $g \in G$ such that $g + E' \subseteq E$. We claim that $n = 0$. Indeed, if $n \geq 1$, then $nE' = \{0\}$ implies $E' \subseteq G[n] \subseteq t(G)$, and so $|g + E'| = |(g + E') \cap E| \leq |(g + t(G)) \cap E| \leq 1$, which yields $|E'| \leq 1$, a contradiction. Therefore, E' is almost 0-torsion. By our hypothesis, $j^{\mathbb{T}}_{\gamma}(E')$ is dense in $\mathbb{T}[0]^{\omega_1 \setminus \gamma}$ (hence in $\mathbb{T}^{\omega_1 \setminus \gamma}$) for some $\gamma \in \omega_1$, and thus $j^{\mathbb{T}}_{\gamma}(g + E')$ is also dense in $\mathbb{T}^{\omega_1 \setminus \gamma}$. This proves that X is finally separable in $\mathbb{T}^{\omega_1}$ by Lemma 9.2. □

Lemma 9.7. *In $M_{\kappa}[\mathbb{G}]$, $\pi(H_{\kappa})$ is hereditarily separable (in the subspace topology generated by $\mathbb{T}^{\omega_1}$).*

Proof. Combine Lemmas 8.1 and 9.6. □

[9] For those readers who are familiar with the classical notion of an HFD set we mention that the above proof actually shows that $G[2]$ is an HFD subset of $\mathbb{T}[2]^{\omega_1}$, and thus hereditary separability of $G[2]$ follows from the classical results about HFD sets, see [33].

10. Making subgroups of $\pi(H_\kappa)$ countably compact

Recall that a point x of a space X is called a *cluster point* of a set $D \subseteq X$ provided that the intersection $U \cap D$ is infinite for every open set U containing x. We will need an alternative description of countably compact spaces: A space X is countably compact provided that every infinite subset of X has a cluster point.

Lemma 10.1. *In the ground model M_κ, let K is a subgroup of H_κ isomorphic to K_n for some $n \in \omega \setminus \{1\}$. Then, in the generic extension $M_\kappa[\mathbb{G}]$, the image $\pi(E)$ of every almost n-torsion set $E \subseteq H_\kappa$ has a cluster point in $\pi(K)$.*

Proof. We will need a piece of notation. For $\gamma \in \omega_1 + 1$, $A \in [\gamma]^{<\omega}$, $\psi : A \to \mathbb{T}$ and $m \geq 1$ we define

$$V_\gamma(A, \psi, m) = \{f \in \mathbb{T}^\gamma : \forall \alpha \in A \ |f(\alpha) - \psi(\alpha)| < 1/m\}.$$

Assume that $E \in M_\kappa[\mathbb{G}]$ is an almost n-torsion subset of H_κ. Since E is countable, $E \in M_\kappa$ by Lemma 7.2. Using Lemmas 6.9 and 6.7 we can find $p \in \mathbb{G}$ such that $E \in \mathcal{E}^p$ (and thus $E \in [H^p]^\omega$) and $\pi^p \restriction_E \colon E \to \mathbb{T}^{\alpha^p}$ is an injection. The latter condition implies that $\pi^p(E)$ is a countable infinite subset of the compact space $\mathbb{T}^{\alpha^p}$, and hence $\pi^p(E)$ must have a cluster point $y \in \mathbb{T}^{\alpha^p}$.

If $n = 0$, then the reader should skip this paragraph and go directly to the next paragraph. Otherwise, we have $n > 1$ and $E \subseteq H_\kappa[n]$, by the definition of almost n-torsion set. Since π^p is a group homomorphism, $\pi^p(E) \subseteq \pi^p(H_\kappa[n]) \subseteq \mathbb{T}[n]^{\alpha^p}$. Since the set $\mathbb{T}[n]^{\alpha^p}$ is closed, it follows that $y \in \mathbb{T}[n]^{\alpha^p}$, and therefore $ny = 0$.

Lemma 6.11 and standard density argument allow us to find $q \in \mathbb{G}$ and $x \in K \cap H^q$ such that $q \leq p$ and $\pi^q(x) \restriction_{\alpha^p} = y$.

If $n = 0$, then the reader should again skip this paragraph. Otherwise $n > 1$, and since $K_n \cong K \subseteq H_\kappa$ and $x \in K$, we have $x \in H_\kappa[n]$. Then $\pi(x) \in \pi(H_\kappa[n]) \subseteq \mathbb{T}[n]^{\omega_1}$ because π is a group homomorphism.

Let us prove that $\pi(x)$ is a cluster point of $\pi(E)$. Suppose that O is an open subset of $\mathbb{T}^{\omega_1}$ such that $\pi(x) \in O$. There exist $A \in [\omega_1]^{<\omega}$, $\psi : A \to \mathbb{T}$ and $m \geq 1$ such that $\pi(x) \in V_{\omega_1}(A, \psi, m) \subseteq O$. Since $\mathbb{T}[0]$ is dense in $\mathbb{T}$ (and $\pi(x) \in \mathbb{T}[n]^{\omega_1}$ when $n > 1$), we can find $\phi : A \to \mathbb{T}[n]$ and $k \geq 1$ such that $\pi(x) \in V_{\omega_1}(A, \phi, k) \subseteq V_{\omega_1}(A, \psi, m)$. Lemma 6.10 and a standard density argument imply the existence of some $r \in \mathbb{G}$ with $r \leq q$ and $A \in [\alpha^r]^{<\omega}$. Define $A^p = A \cap \alpha^p$, $\phi^p = \phi \restriction_{A^p}$, $A^r = A \cap (\alpha^r \setminus \alpha^p) = A \setminus A^p$ and $\phi^r = \phi \restriction_{A^r}$. Since $\pi(x) \in V_{\omega_1}(A, \phi, k)$, it follows that $y = \pi^p(x) \restriction_{\alpha^p} \in V_{\alpha^p}(A^p, \phi^p, k)$. Since y is a cluster point of $\pi^p(E)$ in $\mathbb{T}^{\alpha^p}$ and the set $V_{\alpha^p}(A^p, \phi^p, k)$ is open in $\mathbb{T}^{\alpha^p}$, the intersection

$$\pi^p(E) \cap V_{\alpha^p}(A^p, \phi^p, k) = \{h \in E : \pi^p(h) \in V_{\alpha^p}(A^p, \phi^p, k)\} = E_{A^p, \phi^p, k, \pi^p}$$

must be infinite as π^p is a monomorphism. Since $E \in \mathcal{E}^p$, we conclude that $E_{A^p, \phi^p, k, \pi^p} \in \mathcal{E}^p$ by condition (v_p). Being an infinite subset of an almost n-torsion set E, $E_{A^p, \phi^p, k, \pi^p}$ is also almost n-torsion. Applying condition (iv^r_p) to $A^r \in [\alpha^r \setminus \alpha^p]^{<\omega}$

$\phi^r \in \mathbb{T}[n]^{A^r}$, k and E_{A^p,ϕ^p,k,π^p}, we conclude that the set

$$(E_{A^p,\phi^p,k,\pi^p})_{A^r,\phi^r,k,\pi^r} = E_{A,\phi,k,\pi^r} = \{h \in E : \pi^r(h) \in V_{\alpha^r}(A,\phi,k)\}$$

is infinite. Observe that

$$\{h \in E : \pi^r(h) \in V_{\alpha^r}(A,\phi,k)\} = \{h \in E : \pi(h) \in V_{\omega_1}(A,\phi,k)\}$$

by Lemma 7.4(ii), and so the intersection $\pi(E) \cap V_{\omega_1}(A,\phi,k)$ is also infinite. Since $V_{\omega_1}(A,\phi,k) \subseteq V_{\omega_1}(A,\psi,m) \subseteq O$, the set $\pi(E) \cap O$ must be infinite as well. We have proved that every open neighbourhood O of $\pi(x)$ has an infinite intersection with $\pi(E)$, which yields that $\pi(x)$ is a cluster point of $\pi(E)$. □

Lemma 10.2. *In the generic extension $M_\kappa[\mathbb{G}]$, let G be an infinite Abelian group of size at most $2^\mathfrak{c}$ satisfying conditions* **PS** *and* **CC**. *Then there exists a monomorphism $\varphi : G \to H_\kappa$ such that the subgroup $\pi(\varphi(G))$ of $\pi(H_\kappa)$ is countably compact.*

Proof. Let K be as in Definition 5.1(iii). Since $|K| = \omega_1 < \omega_2 \leq \kappa = r(H_\kappa)$ and $|K| \leq \kappa = r_p(H_\kappa)$ for every $p \in \mathbf{P}$, we can apply Lemma 3.17 (with $G = K$, $G' = \{0\}$, $H = H_\kappa$ and ϕ' the trivial monomorphism) in the ground model M to fix a ground model monomorphism $\theta : K \to H_\kappa$.

In $M_\kappa[\mathbb{G}]$, let G be a group satisfying the assumptions of our lemma. Then $\mathfrak{R}(G) \in M_\kappa[\mathbb{G}]$ is a countable set, so Lemma 7.2 yields $\mathfrak{R}(G) \in M_\kappa$. Since the group $K(G)$ is completely determined by the set $\mathfrak{R}(G) \in M_\kappa$, we have $K(G) \in M_\kappa$. Recall that $K(G)$ is a subgroup of K, and so both $\theta(K(G))$ and $\phi = \theta \restriction_{K(G)} : K(G) \to H_\kappa$ are well-defined. Since θ is a monomorphism, so is ϕ. Note that $|G| \leq 2^\mathfrak{c} = \kappa$ by Lemma 7.6, so we can apply Lemma 5.9 in $M_\kappa[\mathbb{G}]$ to fix a monomorphism $\varphi : G \to H_\kappa$ such that $\theta(K(G)) = \phi(K(G)) \subseteq \varphi(G)$.

Let $S \in M_\kappa[\mathbb{G}]$ be an infinite subset of $\varphi(G)$. We are going to prove that $\pi(S)$ has a cluster point in $\pi(\varphi(G))$. According to Lemma 4.5, there exist $n \in \omega \setminus \{1\}$, $E \in [\varphi(G)]^\omega$ and $h \in \varphi(G)$ so that $h + E \subseteq S$ and E is almost n-torsion in $\varphi(G)$. It follows that $n \in \mathfrak{R}(\varphi(G)) = \mathfrak{R}(G)$, where the equality holds because φ is a monomorphism. Combining Lemmas 5.4 and 5.5, we conclude that $K_n \subseteq K(G)$, and hence $\theta(K_n) \subseteq \theta(K(G)) \subseteq \varphi(G)$. Since $\varphi(G) \subseteq H_\kappa$, Lemma 4.3 allows us to assume that the set E is almost n-torsion in H_κ. Note that $\theta(K_n)$ is a subgroup of H_κ in the ground model which is isomorphic to K_n. Since $E \in M_\kappa$ by Lemma 7.2, Lemma 10.1 allows us to conclude that $\pi(E)$ has a cluster point in $\pi(\theta(K_n)) \subseteq \pi(\varphi(G))$. Then the set $\pi(h + E) = \pi(h) + \pi(E)$ has a cluster point in $\pi(h) + \pi(\varphi(G)) = \pi(\varphi(G))$. (The last equality holds because $h \in \varphi(G)$.) Since $h + E \subseteq S$, $\pi(S) \supseteq \pi(h + E)$ also has a cluster point in $\pi(\varphi(G))$. □

11. Proofs of theorems from Section 2

In this section we provide proofs of theorems left without proof in Section 2. A slight peculiarity of the order in which we choose to provide our proofs should perhaps be mentioned. Namely, we prove Theorem 2.7 before proving Theorem 2.6,

and we give the proof of Theorem 2.28 before that of Theorem 2.27. This is done out of necessity, since we use Theorems 2.7 and 2.28 in our proofs of Theorems 2.6 and 2.27, respectively. The reader should be assured that this does not lead to a circular argument.

Proof of Theorem 2.1: We work in $M_\kappa[\mathbb{G}]$. Implications (iii)$\to$(ii)$\to$(i)$\to$(iv) are trivial. Let us prove the implication (iv)$\to$(iii). Suppose that, in $M_\kappa[\mathbb{G}]$, G is an Abelian group of size at most $2^{\mathfrak{c}}$. Since $|G| \leq 2^{\mathfrak{c}} = \kappa = r(H_\kappa) = r_p(H_\kappa)$ for each $p \in \mathbf{P}$, Lemma 3.18 yields the existence of a monomorphism $\phi : G \to H_\kappa$. Then $\pi(\phi(G))$ is a subgroup of $\pi(H_\kappa)$ algebraically isomorphic to G (Lemma 7.5). Consider G with the topology that its isomorphic image $\pi(\phi(G))$ inherits from $\pi(H_\kappa)$. Since $\pi(H_\kappa)$ is a subgroup of the compact group $\mathbb{T}^{\omega_1}$, the group G with this topology is precompact. Since $\pi(H_\kappa)$ is hereditarily separable (Lemma 9.7) and does not contain non-trivial convergent sequences (Lemma 8.2), G has the same properties.

Proof of Theorem 2.2: Clearly (ii)$\to$(i).

(i)$\to$(iii). Let G be a hereditarily separable group. Since there are no S-spaces, G must be hereditarily Lindelöf, and so $|G| \leq \mathfrak{c}$ [1].

(iii)$\to$(ii). Let G be an Abelian group of size at most $\mathfrak{c}$. Since $\mathbb{T}^\omega$ is a divisible group and $|G| \leq \mathfrak{c} = r(\mathbb{T}^\omega) = r_p(\mathbb{T}^\omega)$ for all $p \in \mathbf{P}$, G is algebraically isomorphic to a subgroup of the compact metric group $\mathbb{T}^\omega$ (Lemma 3.18).

Proof of Theorem 2.7: Clearly (iii)$\to$(ii)$\to$(i) holds in ZFC. The implication (i)$\to$(iv) follows, again in ZFC, from [43] and Lemma 2.5(ii).

It remains only to prove that (iv)$\to$(iii) in $M_\kappa[\mathbb{G}]$. Suppose that, in $M_\kappa[\mathbb{G}]$, G is an Abelian group of size at most $2^{\mathfrak{c}}$ satisfying both **PS** and **CC**. According to Lemma 10.2, there exists, in $M_\kappa[\mathbb{G}]$, a monomorphism $\phi : G \to H_\kappa$ such that $\pi(\phi(G))$ is countably compact. Since $\pi(H_\kappa)$ is hereditarily separable (Lemma 9.7) and does not contain non-trivial convergent sequences (Lemma 8.2), its subgroup $\pi(\phi(G))$ has the same properties. It remains only to note that $\pi(\phi(G))$ is algebraically isomorphic to G.

Proof of Theorem 2.6: Clearly (iii)$\to$(ii)$\to$(i) holds in ZFC. The implication (i)$\to$(iv) follows, again in ZFC, from [43] and Lemma 2.5(i).

It remains only to prove that (iv)$\to$(iii) in $M_\kappa[\mathbb{G}]$. Suppose that, in $M_\kappa[\mathbb{G}]$, G is an Abelian group of size at most $2^{\mathfrak{c}}$ satisfying both **PS** and **tCC**. We need to consider two cases.

Case 1. G is not torsion. In the ground model M, fix a monomorphism $\phi : K_0 \to H_\kappa$. In the generic extension $M_\kappa[\mathbb{G}]$, apply Lemma 10.2(i) to find a monomorphism $\varphi : G \to H_\kappa$ such that $\phi(K_0) \subseteq \varphi(G)$. Therefore, in order to show pseudocompactness of $\pi(\varphi(G))$, it suffices, by Lemma 4 of [21], to establish that the following property holds in $M_\kappa[\mathbb{G}]$: For every $\beta \in \omega_1$ and for every $y \in \mathbb{T}^\beta$ there exists $x \in \phi(K_0)$ such that $\pi(x) \restriction_\beta = y$. This property, however, immediately follows from

the fact that $\phi(K_0)$ is a subgroup of H_κ in the ground model M_κ algebraically isomorphic to $\mathbb{Z}^{(\omega_1)}$ and Lemma 6.11. Since $\pi(H_\kappa)$ is hereditarily separable (Lemma 9.7) and does not contain non-trivial convergent sequences (Lemma 8.2), its subgroup $\pi(\phi(G))$ has the same properties. It remains only to note that $\pi(\phi(G))$ is algebraically isomorphic to G.

Case 2. G is torsion. Then G satisfies **CC** by item (vi) of Lemma 2.4. Since G also satisfies **PS**, we can apply Theorem 2.7 (which has been proved by now) to get a hereditarily separable countably compact (hence, pseudocompact) group topology without non-trivial convergent sequences on G.

Proof of Theorem 2.8: Clearly (iii)→(ii)→(i) holds in ZFC. To see that (i)→(iii) if there are no S-spaces, note that in a model of ZFC without S-spaces, a hereditarily separable group G must be hereditarily Lindelöf, and thus G is compact and first countable provided that G is pseudocompact. Finally, first countable groups are metrizable.

Proof of Theorems 2.11 and 2.12: The implication (ii)→(i) in both theorems is trivial. The implication (i)→(iii) in Theorems 2.11 and 2.12 follows from the implication (i)→(iv) of Theorems 2.6 and 2.7 respectively. To prove the remaining implication (iii)→(ii) in both theorems we note that, according to Case 1 in the proof of Theorem 2.6, there exists a monomorphism $\varphi : G \to H_\kappa$ such that $\pi(\varphi(G))$ is a dense pseudocompact subgroup of $\mathbb{T}^{\omega_1}$. Since $\mathbb{T}^{\omega_1}$ is connected and locally connected, so is $\pi(\varphi(G))$ (see, for example, [14, Fact 2.10]).

Proof of Theorem 2.14: Let us consider the following additional condition (iv): $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **CC**.

(i) $\leftrightarrow$ (iv) has been proved in Theorem 2.7.

(i) $\to$ (ii). Let G be a non-trivial divisible Abelian group that admits a (separable) countably compact group topology. Recall that torsion pseudocompact groups are zero-dimensional [10] and pseudocompact divisible groups are connected [59]. Since G is non-trivial, it follows that G is non-torsion. Now (ii) follows from the implication (i) $\to$ (ii) of Corollary 2.12.

(ii) $\to$ (i) is trivial.

(iv) $\to$ (iii). Since a non-trivial divisible group cannot be a bounded torsion group, the inequality $r(G) \geq \mathfrak{c}$ follows from the definition of **PS**. If p is a prime number such that $r_p(G)$ is infinite, then $r_p(G) = |G[p]|$ and $|G[p]| = |1 \cdot G[p]| \geq \mathfrak{c}$ by **CC**.

(iii) $\to$ (iv). Since $r(G) \geq \mathfrak{c}$, **PS** holds. It remains only to prove that G satisfies **CC**. Let $m \geq 1$ and $n \geq 1$ be arbitrary integers. Assume, without loss of generality, that $mG[n]$ is infinite. This obviously yields $n > 1$, so let $n = p_1^{r_1} \dots p_s^{r_k}$ be the factorization of n with distinct primes $p_1, \dots, p_k$. Then $G[n] = \bigoplus_{i=1}^{k} G[p_i^{r_i}]$, and hence $mG[n] = \bigoplus_{i=1}^{k} mG[p_i^{r_i}]$. Since $mG[n]$ is infinite, $mG[p_j^{r_j}]$ must be infinite for some $j = 1, \dots, k$. Then $\rho = r_{p_j}(G) \geq \mathfrak{c}$ by (iii). Since G is divisible, the

p_j-torsion part $G_{p_j} = \bigcup_{n\in\omega} G[p_j^n]$ of G is divisible too[10], and hence $G_{p_j} \cong \mathbb{Z}(p_j^\infty)^{(\rho)}$ by [24, Theorem 23.1]. Therefore, $G[p_j^{r_j}] \cong \mathbb{Z}(p_j^{r_j})^{(\rho)}$. Since the group $mG[p_j^{r_j}]$ is infinite, m does not divide $p_j^{r_j}$. In particular, $m\mathbb{Z}(p_j^{r_j}) \neq \{0\}$. This proves that $mG[p_j^{r_j}] \cong \left(m\mathbb{Z}(p_j^{r_j})\right)^{(\rho)}$ has size ρ, and so $|mG[n]| \geq |mG[p_j^{r_j}] = \rho \geq \mathfrak{c}$.

Proof of Theorem 2.22: The "only if" part is proved in Lemma 2.5(i). Let us prove the "if" part. According to [12], an infinite cardinal τ is called *admissible* provided that there exists a pseudocompact group of size τ (see also [14, Definition 3.1(i)]). We need to consider two cases.

Case 1. G is not torsion. Then **PS** implies $r(G) \geq \mathfrak{c}$, and therefore we have $\mathfrak{c} = \mathfrak{c}^\omega \leq r(G) \leq |G| \leq 2^\mathfrak{c}$. Now the existence of a pseudocompact group topology on G follows from [12, Theorem 6.4] or [14, Corollary 7.4] (where one needs to take $\tau = \mathfrak{c}$).

Case 2. G is torsion. By item (i) of Lemma 2.4, G is a bounded torsion group, and thus $G[n] = G$ for some natural number $n \geq 1$. Item (vi) of Lemma 2.4 implies that G satisfies **CC**. Therefore, for every $m \in \omega \setminus \{0\}$, the group $mG[n]$ is either finite or satisfies the inequality $\mathfrak{c} \leq |mG[n]| \leq |G| \leq 2^\mathfrak{c}$. In the latter case, the cardinal $|mG[n]|$ is admissible by items (i) and (ii) of [14, Lemma 3.4]. Now the existence of a pseudocompact group topology on $G[n]$ $(= G)$ follows from the implication (d) $\to$ (a) of [14, Theorem 6.2].

Proof of Theorem 2.28: Our theorem follows immediately from the equivalence of items (iii) and (iv) of Theorem 2.6 and the following simple observations:

(1) An S-space must be infinite, and an infinite pseudocompact group has size at least $\mathfrak{c}$ [20, Proposition 1.3 (a)].

(2) An infinite pseudocompact group G without non-trivial convergent sequences is not Lindelöf.

To get (2) note that G cannot be compact because infinite compact groups contain non-trivial convergent sequences (see item (ii) of Fact 1.2). Since pseudocompact Lindelöf spaces are compact, G cannot be Lindelöf.

Proof of Theorem 2.27: The implication (ii)$\to$(i) is trivial. To prove (i)$\to$(iii), let G be an S-space. Then $|G| \leq 2^\mathfrak{c}$ because G is separable [43]. Since G is not Lindelöf, G cannot be countable, and thus $|G| > \omega$. Since $\omega_1 = \mathfrak{c}$ holds in $M_\kappa[\mathbb{G}]$ (Lemma 7.6), one gets $|G| \geq \mathfrak{c}$.

Let us prove that (iii)$\to$(ii). Let G be an Abelian group in $M_\kappa[\mathbb{G}]$ such that $\mathfrak{c} \leq |G| \leq 2^\mathfrak{c}$. If $r(G) \geq \mathfrak{c}$, then G satisfies **PS** and **tCC** (see item (v) of Lemma 2.4), and so we can apply Theorem 2.28 (that has been proved by now) to get a pseudocompact group topology on G that makes G into an S-space. Since pseudocompact groups

[10]Recall that a subgroup H of an Abelian group G is *pure* (in G) provided that $nH = nG \cap H$ for every $n \in \omega$. Note that G_{p_j} is a pure subgroup of $t(G)$ (being its direct summand). Since $t(G)$ is a pure subgroup of G and purity is transitive, it follows that G_{p_j} is a pure subgroup of G. Now it remains only to note that every pure subgroup of a divisible Abelian group is divisible.

are precompact [8], we are done in this case. It remains only to consider the case $r(G) < \mathfrak{c}$. Since $\mathfrak{c} \leq |G| = \max\{r(G), \sup_{p\in\mathbf{P}} r_p(G)\}$ and $\mathrm{cf}(\mathfrak{c}) > \omega$, we can fix $p \in \mathbf{P}$ with $r_p(G) \geq \mathfrak{c}$. Therefore $|G[p]| \geq \mathfrak{c}$ and $G[p]$ satisfies **CC**. Since $G[p]$ is a bounded torsion group, it satisfies **PS** by Lemma 2.4(i). Finally, $|G[p]| \leq |G| \leq 2^{\mathfrak{c}}$. Applying Lemma 10.2 to the group $G[p]$ we can find, in $M_\kappa[\mathbb{G}]$, a monomorphism $\phi : G[p] \to H_\kappa$ such that $\pi(\phi(G[p]))$ is countably compact. In $M_\kappa[\mathbb{G}]$, the following inequalities hold: (i) $|G[p]| \leq 2^{\mathfrak{c}} = \kappa = r(H_\kappa)$, and (ii) $|G[p]| \leq 2^{\mathfrak{c}} = \kappa = r_q(H_\kappa)$ for every $q \in \mathbf{P}$. (We have applied here Lemma 7.6.) Since H_κ is divisible, we can apply Lemma 3.17 (with $G[p]$ as G' and H_κ as H) in $M_\kappa[\mathbb{G}]$ to get a monomorphism $\varphi : G \to H_\kappa$ extending ϕ. Note that $\pi(\varphi(G))$ is algebraically isomorphic to G. Being a subgroup of the compact group $\mathbb{T}^{\omega_1}$, $\pi(\varphi(G))$ is totally bounded. Since $\pi(H_\kappa)$ is hereditarily separable (Lemma 9.7) and does not contain non-trivial convergent sequences (Lemma 8.2), its subgroup $\pi(\varphi(G))$ has the same properties. To prove that $\pi(\varphi(G))$ is an S-space, it remains only to check that $\pi(\varphi(G))$ is not Lindelöf. Observe that $\pi(\phi(G[p])) = \pi(\varphi(G[p])) = \pi(\varphi(G))[p]$, and therefore $\pi(\phi(G[p]))$ is a (unconditionally) closed subgroup of the group $\pi(\varphi(G))$. Hence, if the latter group were Lindelöf, its closed subgroup $\pi(\phi(G[p]))$ would be Lindelöf as well. Being countably compact, $\pi(\phi(G[p]))$ must be compact. Since $\pi(\phi(G[p]))$ does not have non-trivial convergent sequences, it must be finite (see item (ii) of Fact 1.2), a contradiction.

Proof of Theorem 2.29: Since countable compactness implies pseudocompactness, the proof of this theorem is similar to that of Theorem 2.28, with reference to Theorem 2.6 replaced by reference to Theorem 2.7.

12. Algebraic structure of compact metric Abelian groups

Recall that an Abelian group G is *reduced* if it does not have non-zero divisible subgroups. We start with a well-known algebraic property of Abelian groups.

Lemma 12.1. *Every Abelian group G admits a unique representation $G = D(G) \oplus R(G)$, where $D(G)$ is the maximal divisible subgroup of G, the subgroup $R(G) \cong G/D(G)$ of G is reduced and*

$$D(G) \cong \mathbb{Q}^{(\mathfrak{s})} \oplus \left(\bigoplus_{p\in\mathbf{P}} \mathbb{Z}(p^\infty)^{(\mathfrak{s}_p)} \right), \tag{8}$$

where $\mathfrak{s}$, $\mathfrak{s}_p$ for $p \in \mathbf{P}$ are suitable cardinals uniquely determined by G.

Proof. According to Corollary 3.3, a divisible subgroup of an Abelian group is always a direct factor, hence we can write $G = D(G) \times R(G) = D(G) \oplus R(G)$, where the subgroup $R(G)$ of G is reduced and $R(G) \cong G/D(G)$. The fact that the divisible part $D(G)$ of G has form (8) can be found in [24, Theorem 23.1]. □

While the description of Abelian groups admitting compact group topology is well-known (see, for example, [30]), the case of *metrizable* compact groups cannot

be found in the literature. This is why we provide a complete self-contained proof of this case in our next theorem.

Theorem 12.2. *(1) An Abelian group G admits a compact (metric) group topology if and only if both its divisible part $D(G)$ and its reduced part $R(G)$ admit a compact (metric) group topology.*

(2) $D(G)$ admits a compact metric group topology if and only if the cardinals $\mathfrak{s}$ and $\mathfrak{s}_p$ ($p \in \mathbf{P}$) from Lemma 12.1 satisfy the following conditions:

(2a) *Either $\mathfrak{s} = 0$ or $\mathfrak{s} = \mathfrak{c}$,*

(2b) *For each $p \in \mathbf{P}$, $\mathfrak{s}_p \leq \mathfrak{s}$ holds, and either $\mathfrak{s}_p$ is finite or $\mathfrak{s}_p = \mathfrak{c}$.*

(3) $R(G)$ admits a compact group topology if and only if

$$R(G) \cong \prod_{p \in \mathbf{P}} \left(\mathbb{Z}_p^{\mathfrak{b}_p} \times \prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\mathfrak{a}_{n,p})} \right). \tag{9}$$

(4) $R(G)$ admits a compact metric group topology if and only if (9) and the cardinals from (9) satisfy the following conditions:

(4a) *$\mathfrak{b}_p \leq \omega$ for every $p \in \mathbf{P}$,*

(4b) *For every $p \in \mathbf{P}$ and each $n \in \omega \setminus \{0\}$, the cardinal $\mathfrak{a}_{n,p}$ is either finite or equal to $\mathfrak{c}$.*

Proof. (1) The "if" part is clear. To prove the "only if" part, assume that G is a compact (metric) group. The connected component $c(G)$ of G is a closed subgroup of G. Therefore, both $c(G)$ (considered as a subspace of G) and the quotient group $G/c(G)$ are compact (metric) groups.

Since $R(G) \cong G/D(G)$, it remains only to prove that $c(G) = D(G)$. It is a well-known fact that a compact Abelian group is connected if and only if it is divisible [30, Theorem 24.25].[11] Hence $c(G)$, being connected, is divisible. By maximality of $D(G)$ we have $c(G) \subseteq D(G)$. The closure H of $D(G)$ in G is a compact group. Let $n \in \omega \setminus \{0\}$. Being the image of the compact set H under the continuous map that sends h to nh, the set nH is compact as well. In particular, nH is closed in G. Note that $D(G) = nD(G) \subseteq nH$ by divisibility of $D(G)$, and hence $H \subseteq nH$. We have proved that $H \subseteq nH$ for every $n \in \omega \setminus \{0\}$, which yields divisibility of H. Applying the result cited in the beginning of this paragraph to the compact group H, we conclude that H is connected, and so $D(G) \subseteq H \subseteq c(G)$.

(2) According to [30, Theorems 25.23, 25.24], $D(G)$ admits a compact metrizable group topology if and only if either $D(G) = \{0\}$ (i.e., $\mathfrak{s} = \mathfrak{s}_p = 0$ for each $p \in \mathbf{P}$) or $\mathfrak{s} = \mathfrak{c}$ and, for each $p \in \mathbf{P}$, the cardinals $\mathfrak{s}_p$ take only finite values or $\mathfrak{c}$.

(3) According to [30, Theorem 25.22], the reduced part $R(G)$ admits a compact group topology if and only if (9) holds for suitable cardinals $\mathfrak{b}_p$ and $\mathfrak{a}_{n,p}$.

(4) To prove the "if" part, assume that (9) holds, where cardinals $\mathfrak{b}_p$ and $\mathfrak{a}_{n,p}$ satisfy conditions (4a) and (4b).

[11] For a comment on the counterpart of this property in the noncompact case see Example 13.14.

Then for $p \in \mathbf{P}$ and $n \in \omega\setminus\{0\}$ the group $\mathbb{Z}(p^n)^{(\mathfrak{a}_{n,p})}$ is either finite, or algebraically isomorphic to $\mathbb{Z}(p^n)^\omega$. In both cases it carries a compact metrizable group topology. The product topology of the group $\mathbb{Z}_p^{\mathfrak{b}_p}$ is also a compact metrizable group topology. Now the product topology on the product in (9) is a compact metrizable group topology for $R(G)$.

Before proceeding with the rest of the proof, recall that a subgroup H of an Abelian group G is *pure* (in G) provided that $nH = nG \cap H$ for every $n \in \omega$.

To prove the "only if" part, assume that $R(G)$ is equipped with a compact metrizable group topology. Since $D(R(G)) = \{0\}$, this topology is totally disconnected [11, Corollary 3.3.9]. Hence the Pontryagin dual X of G is a torsion countable group [30, Theorems 24.15, 24.26]. Let $X = \bigoplus_{p\in\mathbf{P}} X_p$, where X_p is the p-torsion subgroup of X. According to [24, §33], X_p admits a *p-basic* subgroup B_p, i.e., a pure subgroup B_p such that X_p/B_p is divisible, so $X_p/B_p \cong \mathbb{Z}(p^\infty)^{(\mathfrak{b}_p)}$, and $B_p = \bigoplus_{n\in\omega\setminus\{0\}} \mathbb{Z}(p^n)^{(\mathfrak{r}_{n,p})}$ with $\mathfrak{b}_p \le \omega$ and $\mathfrak{r}_{n,p} \le \omega$. Consequently, the compact Pontryagin dual K_p of X_p has a closed subgroup $N_p = A(B_p)$ (the annihilator of B_p) such that N_p is isomorphic to the dual of the divisible quotient X_p/B_p, hence

$$N_p \cong \mathbb{Z}_p^{\mathfrak{b}_p} \quad \text{and} \quad K_p/N_p \cong \prod_{n\in\omega\setminus\{0\}} \mathbb{Z}(p^n)^{\mathfrak{r}_{n,p}}, \tag{10}$$

the group K_p/N_p being isomorphic to the dual of B_p.

Claim 1. For each $p \in \mathbf{P}$, N_p is a pure subgroup of K_p.

Proof. For $n \in \omega$ let $B = \{x \in X_p : nx \in B_p\}$, so that $nK_p = A(B)$ is the annihilator of B in K_p. By the purity of B_p one can easily deduce that $B = X_p[n] + B_p$. Indeed, if $nx \in B_p$ for some $x \in X_p$, then $nx \in B_p \cap nX_p = nB_p$, so $nx = nb$ for some $b \in B_p$ and consequently $x - b \in X_p[n]$. Taking annihilators we get

$$nN_p = nA(B_p) = A(B) = A(X_p[n] + B_p) = A(X_p[n]) \cap A(B_p) = nK_p \cap N_p.$$

This proves that N_p is a pure subgroup of K_p. □

According to [30, Theorem 25.21] and the above claim, the subgroup N_p of K_p is a direct factor of K_p, i.e., $K_p \cong N_p \times K_p/B_p$. Since $R(G) \cong \prod_{p\in\mathbf{P}} K_p$, (10) yields (9), since $\mathbb{Z}(p^n)^{\mathfrak{r}_{n,p}} \cong \mathbb{Z}(p^n)^{(\mathfrak{a}_{n,p})}$, where $\mathfrak{a}_{n,p} = \mathfrak{r}_{n,p}$ if the latter cardinal is finite, otherwise $\mathfrak{a}_{n,p} = 2^{\mathfrak{r}_{n,p}} = \mathfrak{c}$ when $\mathfrak{r}_{n,p} = \omega$. □

As an easy application of the above theorem, we can see that some well-known groups do not admit compact group topologies.

Example 12.3. *Neither the Specker group $\mathbb{Z}^\omega$, nor any free Abelian group admit a compact group topology.* Indeed, let G be either the Specker group or a free Abelian group. Assume that G has a compact group topology. Since $D(G) = \{0\}$, from Theorem 12.1 it follows that $G \cong R(G)$, and therefore G must have the form (9) in view of item (3) of Theorem 12.2. Since G is torsion free, all $\mathfrak{a}_{n,p}$ are zero. Hence $G = \prod_{p\in\mathbf{P}} \mathbb{Z}_p^{\mathfrak{b}_p}$. Assume that $\mathfrak{b}_p \neq 0$ for some prime p. Then, for every prime $q \neq p$,

the subgroup $H = \mathbb{Z}_p^{\mathfrak{b}_p}$ of G is *q-divisible*, i.e., $qH = H \neq 0$. On the other hand, it is easy to see that $\bigcap_{n\in\omega} q^n G = \{0\}$, a contradiction since this intersection must contain the non-zero subgroup H.

Example 12.4. Let G be either the Specker group $G = \mathbb{Z}^\omega$ or the free Abelian group of size $\mathfrak{c}$. Then *the existence of a hereditarily separable pseudocompact group topology on G is both consistent with and independent of ZFC*. Indeed, our previous example and Theorem 2.8 imply that G does not admit a hereditarily separable pseudocompact group topology in any model of ZFC in which there are no S-spaces. On the other hand, according to Theorem 2.11, in the model $M_\kappa[\mathbb{G}]$, the group G admits a hereditarily separable, pseudocompact, connected and locally connected group topology without non-trivial convergent sequences.

13. Final remarks and open problems

The reader may wonder if weaker versions of our main results, with "hereditarily separable" weakened to "separable", can be proved in ZFC. The following three theorems providing a positive answer to this question are particular cases of general results from [16].

Theorem 13.1. *The following conditions are equivalent for any Abelian group G:*
(i) G has a separable group topology,
(ii) G has a separable precompact group topology,
(iii) $|G| \leq 2^{\mathfrak{c}}$.

Theorem 13.2. *The following conditions are equivalent for any Abelian group G:*
(i) G admits a separable pseudocompact group topology,
(ii) $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **tCC**.

Theorem 13.3. *The following conditions are equivalent for any Abelian group G:*
(i) G admits a separable connected precompact group topology,
(ii) G is non-torsion and admits a separable pseudocompact group topology,
(iii) G is a non-torsion group satisfying both $|G| \leq 2^{\mathfrak{c}}$ and **PS**,
(iv) $\mathfrak{c} \leq r(G) \leq |G| \leq 2^{\mathfrak{c}}$.

The lack of any ZFC results about separable countably compact topologies on Abelian groups justifies our next problem:

Problem 13.4. Describe in ZFC the algebraic structure of separable countably compact Abelian groups.

The next question provides a natural hypothesis for the solution of the above problem:

Question 13.5. Is it true in ZFC that an Abelian group G admits a separable countably compact group topology if and only if $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **CC**?

Question 13.6. Is it true in ZFC that an Abelian group G of size at most $2^{\mathfrak{c}}$ admits a countably compact group topology if and only if G satisfies both **PS** and **CC**?

Our Theorem 2.7 gives a strong positive *consistent* answer to both Questions 13.5 and 13.6. We hope that an answer to these two questions will not involve set-theoretic complications described in Example 12.4.

Question 13.7. (i) Is it true in ZFC that the Specker group $\mathbb{Z}^\omega$ admits a countably compact group topology?

(ii) Does $\mathbb{Z}^\omega$ has a separable countably compact group topology in ZFC?

(iii) In ZFC, does $\mathbb{Z}^\omega$ admit a (separable) connected, locally connected group topology?

(iv) In ZFC, does $\mathbb{Z}^\omega$ admit a countably compact separable group topology without non-trivial convergent sequences?

Our Corollary 2.13 gives a strong consistent positive answer to (all items of) the above question. The reader may also want to consult Example 12.4 for relevant independence results.

Our next two questions are motivated by Corollary 2.20.

Question 13.8. Is there a torsion Abelian group that admits a pseudocompact group topology but does not admit a countably compact group topology?

Question 13.9. Does there exist a torsion-free Abelian group that admits a pseudocompact group topology but does not admit a countably compact group topology?

Recall that an Abelian group G is called *algebraically compact* if there exists an Abelian group H such that the direct sum $G \oplus H$ admits a compact group topology, i.e. if G is a direct summand of some compact group. (More precisely, Kaplansky [35] introduced algebraically compact groups via several equivalent properties, including this one). Algebraically compact groups form a relatively narrow subclass of Abelian groups (for example, the integers $\mathbb{Z}$ are not algebraically compact) that plays a prominent role in the theory of infinite (abstract) Abelian groups. It has been shown in [14, Theorem 8.15] that every Abelian group G is "*algebraically pseudocompact*" in the sense that one can find an Abelian group H such that $G \oplus H$ admits a *pseudocompact* group topology. This makes it natural to wonder whether this result could be strengthened to show that every Abelian group is "algebraically countably compact":

Question 13.10. Is every Abelian group a direct summand of an Abelian group that admits a countably compact group topology?

Our next theorem provides a positive consistent answer to this question for Abelian groups of small size:

Theorem 13.11. *In $M_\kappa[\mathbb{G}]$, for every Abelian group G of size at most $2^{\mathfrak{c}}$ there exists an Abelian group H such that $G \oplus H$ admits a (hereditarily separable, connected,*

locally connected) countably compact group topology (without non-trivial convergent sequences).

Proof. Let $H = H_\kappa$ and $A = G \oplus H$. Since $r(A) \geq r(H) = 2^{\mathfrak{c}} > \mathfrak{c}$, A satisfies **PS**. For every pair of integers $m \geq 1$ and $n \geq 1$, one has $mA[n] = mG[n] \oplus mH[n]$, and so $|mA[n]| \geq |mH[n]| = 2^{\mathfrak{c}} > \mathfrak{c}$. Thus A satisfies **CC**. By Theorem 2.11 A admits a (hereditarily separable, connected, locally connected) countably compact group topology (without non-trivial convergent sequences). □

It was proved in [18, Corollary 5.3] that a divisible Abelian group of size $\mathfrak{c}$ admits a countably compact group topology if and only if it admits a compact group topology. Item (i) of our next example demonstrates that this equivalence no longer holds for divisible Abelian groups of size bigger than $\mathfrak{c}$.

Example 13.12. Let p be a prime number and σ, τ cardinals satisfying $\mathfrak{c} \leq \sigma < \tau \leq 2^{\mathfrak{c}}$, and let $G_{\sigma,\tau} = \mathbb{Z}(p^\infty)^{(\tau)} \oplus \mathbb{Q}^{(\sigma)}$.

(i) *In $M_\kappa[\mathbb{G}]$, the divisible group $G_{\sigma,\tau}$ admits a (hereditarily separable, connected, locally connected) countably compact group topology (without non-trivial convergent sequences) but cannot be equipped with any compact group topology.* Indeed, the existence of the required countably compact group topology on $G_{\sigma,\tau}$ follows from Theorem 2.14. On the other hand, since $r(G) = \sigma < \tau = r_p(G)$, $G_{\sigma,\tau}$ does not admit a compact group topology by [30, Theorem 25.23].

(ii) *In any model of ZFC without S-spaces, $G_{\sigma,\tau}$ does not admit a hereditarily separable pseudocompact group topology.* Indeed, as was noted above, $G_{\sigma,\tau}$ does not admit a compact group topology, and now the conclusion follows from Theorem 2.8.

Combining (i) and (ii), we get

(iii) *The existence of a hereditarily separable countably compact (or pseudocompact) group topology on $G_{\sigma,\tau}$ is both consistent with and independent of ZFC.*

Problem 13.13. In ZFC, give an example of a divisible Abelian group that admits a countably compact group topology but does not admit a compact group topology.

Recall that divisible pseudocompact groups are connected [59]. We show in our next example that (even) countably compact connected groups need not be divisible.

Example 13.14. *A connected countably compact Abelian group that is not divisible.* Take $K = \mathbb{T}^{\mathfrak{c}}$ and let $G = \{x \in \mathbb{T}^{\mathfrak{c}} : |\{\alpha \in \mathfrak{c} : x(\alpha) \neq 0\}| \leq \omega\}$ be the Σ-product of $\mathfrak{c}$-many copies of $\mathbb{T}$ considered as a subgroup of K. Let a be the only element of $\mathbb{T}$ of order 2 and let $\mathbf{a} \in K$ be the element having all coordinates equal to a. Then $C = \langle\!\langle \mathbf{a} \rangle\!\rangle \cong \mathbb{Z}(2)$ trivially meets G, hence $H = G + C = G \oplus C$ is not divisible. On the other hand, H is countably compact since every countable subset of H is contained in a compact subgroup of H. As a dense countably compact subgroup of the connected group K, the group H is connected too (see, for example, [14, Fact 2.10]).

We finish this paper with a series of questions related to the existence of convergence sequences in compact-like groups.

Our Corollary 2.16 both motivates our next question and demonstrates that the positive answer to it for Abelian groups of size at most $2^{\mathfrak{c}}$ is consistent with ZFC.

Question 13.15. (i) Does every pseudocompact (Abelian) group admit a pseudocompact group topology without non-trivial convergent sequences?

(ii) Does every countably compact (Abelian) group admit a countably compact group topology without non-trivial convergent sequences?

The next question, going in the opposite direction, may be considered as a "countably compact heir" of Fact 1.2(i) that still has a chance of positive answer in ZFC.

Question 13.16. Let G be an infinite countably compact group. Does G have a countably compact group topology that contains a non-trivial convergent sequence?

The pseudocompact variant of this question seems to be open as well.

Question 13.17. Let G be an infinite pseudocompact group. Does G have a pseudocompact group topology that contains a non-trivial convergent sequence?

An infinite symmetric group $S(X)$ and a free group $F(X)$ do not admit *any* countably compact group topology (Proposition 1.3 and [12, Theorem 4.7]; see also [14, Corollary 5.14]). These two examples are "highly non-commutative" in nature. Since it appears to be so hard to get countably compact group topologies on "highly non-commutative groups", one might hope that when such groups *do* admit a countably compact group topology, this topology must necessarily have a non-trivial convergent sequence.

Recall that the *derived subgroup* G' of a group G is the smallest subgroup of G that contains the set $\{xyx^{-1}y^{-1} : x, y \in G\}$. Obviously, G is Abelian if and only if G' is the trivial subgroup of G. A group G is called a *perfect group* if it satisfies $G' = G$.

Question 13.18. If G is an infinite countably compact group satisfying $G' = G$, must G have a non-trivial convergent sequence?

Question 13.19. Let G be an infinite countably compact group without open Abelian subgroups. Does G have a non-trivial convergent sequence?

At first look the requirement "without open Abelian subgroups" appears to be a somewhat poor approximation of an the intuitive notion of "highly non-commutative group". Indeed, the following version of Question 13.19 seems to be a much better choice for capturing that notion: *If G is an infinite countably compact group with trivial center $Z(G) = \{g \in G : xg = gx \text{ for each } x \in G\}$, must G have a non-trivial convergent sequence?* Unfortunately, Example 13.21 below shows that this version has a consistent negative answer. In order to construct a counter-example, we will need a standard construction of a semidirect product.

Identify the cyclic group $\mathbb{Z}(2)$ with the multiplicative group $\{1,-1\}$. Let A be an Abelian group. The *semidirect product* $A \rtimes \mathbb{Z}(2)$ of A and $\mathbb{Z}(2)$ with respect to the action of $\mathbb{Z}(2)$ on A given by $a \mapsto -a$ is defined as the Cartesian product $A \times \mathbb{Z}(2)$ with operation $(a,x)\cdot(a',x') := (a+xa', xx')$. We identify A with the subgroup $A \times \{1\}$ of $A \rtimes \mathbb{Z}(2)$ via the map $a \mapsto (a,1)$. If A is a topological group, then $A \rtimes \mathbb{Z}(2)$ equipped with the product topology is a topological group. (Here $\mathbb{Z}(2)$ carries the discrete topology.)

Proposition 13.20. *Let A be a countably compact Abelian group and let $G = A \rtimes \mathbb{Z}(2)$ be the semidirect product with respect to the action $a \mapsto -a$ of $\mathbb{Z}(2)$ on A. Then the product topology makes G into a countably compact topological group containing A as an open (normal) subgroup of index 2. Moreover,*

(a) G is Abelian if and only if A is Boolean;
(b) G has trivial center if and only if $r_2(A) = 0$;
(c) the derived subgroup G' of G coincides with $2A \times \{1\}$;
(d) G has non-trivial convergent sequence if and only if A does;
(e) G is (hereditarily) separable if and only if A is (hereditarily) separable;
(f) G is locally connected if and only if A is locally connected;
(g) G is totally disconnected if and only if A is totally disconnected.

Proof. (a) follows from the fact that the action of $\mathbb{Z}(2)$ on A is trivial if and only if A is a Boolean group.

To verify (b) note that when A is not Boolean, then the center of G is precisely $A[2] \times \{1\}$.

(c) Under the identification of the subgroup $A \times \{1\}$ of G with A, the quotient group $G/2A \cong \mathbb{Z}(2) \rtimes \mathbb{Z}(2) \cong \mathbb{Z}(2) \times \mathbb{Z}(2)$ is Abelian by item (b). This yields $G' \subseteq 2A$. To prove the opposite inclusion take an $a \in A$ and note that $(2a,1) = z^{-1}\cdot(a,1)^{-1}\cdot z\cdot(a,1) \in G'$, where $z = (0,-1)$.

(d)-(g) easily follow from the fact that A is an open subgroup of G. □

Example 13.21. Let A be any Abelian group without non-trivial convergent sequences such that $r_2(A) = 0$. Then $G_A = A \rtimes \mathbb{Z}(2)$ is a countably compact group with trivial center and without non-trivial convergent sequences (items (b) and (d) of Proposition 13.20).

(i) If τ is a cardinal with $\mathfrak{c} \le \tau \le 2^{\mathfrak{c}}$ and A is the group $\mathbb{Z}^{(\tau)}$ equipped with the topology from item (ii) of Corollary 2.13, then Proposition 13.20 yields that G_A is a *hereditarily separable countably compact, locally connected group with trivial center and without non-trivial convergent sequences*.

(ii) If $p > 2$ is a prime, τ is a cardinal with $\mathfrak{c} \le \tau \le 2^{\mathfrak{c}}$ and the group $A = \mathbb{Z}(p)^{(\tau)}$ is equipped with the topology from item (ii) of Corollary 2.10, then Proposition 13.20 yields that G_A is a *hereditarily separable countably compact zero-dimensional group with trivial center and without non-trivial convergent sequences*. (Zero-dimensionality follows from [10].)

Example 13.22. Let A be any countably compact Abelian group without non-trivial convergent sequences such that $2A = A$. According to items (c), (d) and (e) of Proposition 13.20, the *group* $G_A = A \rtimes \mathbb{Z}(2)$ *satisfies* $G_A/G'_A \cong \mathbb{Z}(2)$, *is both hereditarily separable and countably compact, and does not have non-trivial convergent sequences.*

Observe that the derived subgroup G'_A of the group G_A from our previous example is "large" in the sense that the quotient $G_A/G'_A \cong \mathbb{Z}(2)$ is very "small", and yet G_A is still very far from being a perfect group. Indeed, the derived subgroup G'_A of G_A is Abelian, and thus G_A itself is meta-Abelian.

Note that all groups of the form G_A in Examples 13.21 and 13.22 have an open Abelian subgroup A of index 2. Therefore they have no negative impact on Question 13.19, and in fact, make it appear now even natural.

Finally, we recall that MA yields a positive answer to both Questions 13.18 and 13.19 for groups of weight $< \mathfrak{c}$ [9].

Remark 13.23. Additional applications of our results from Section 2 can be found in the forthcoming paper [15].

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