Remarks on two problems by M. Laczkovich on functions with Borel measurable differences

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Abstract

We consider Problems 2 and 3 in [3] asked by M.Laczkovich concerning the difference property of Borel measurable functions. We show that the axiom of determinacy implies affirmative answer to Problem 2 (Theorem 2) and that Problem 3 is settled affirmatively for all infinite order Baire classes (Theorem 1.)

Introduction

In this article, we consider Problems 2 and 3 asked by M.Laczkovich in [3]. We are, however, not able to give here the final answer to either. Instead, we show some evidence that leads us to conjecture that affirmative answers to both problems are possible (at least, in the sense of consistency.)

Laczkovich's paper [3] pursues questions about the difference property of real functions: *What can you say about function $f : \mathbb{R} \to \mathbb{R}$ when the differences $f(x+h) - f(x)$ are known to be in a given class for all $h \in \mathbb{R}$?* This pursuit was initiated by N.G. de Bruijn who proved that if the difference $f(x + h) - f(x)$ is a continuous functions of x for each fixed $h \in \mathbb{R}$, then $f = g + A$ with continuous g and additive A (that is to say, A satisfies the functional equation $A(x + y) = A(x) + A(y)$.) It was then asked whether similar result holds for other classes of real functions.

In [3], Laczkovich considered the class of Lebesgue measurable functions. Now it is known that whether de Bruijn's result transfers to the context of Lebesgue measurable functions is independent of the usual axioms of set theory. See [4]. Along the line of pursuit, Laczkovich left three questions open, first of which he solved by himself shortly after that. Remaining two questions were the following

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Problem 2. *Suppose that $f(x+h)-f(x)$ is Borel measurable for every $h \in \mathbb{R}$. Is it true that the functions $f(x+h)-f(x)$ belong to the* same *Baire class of order $\alpha < \omega_1$?*

Problem 3. *Let f be Borel measurable and suppose that $f(x+h)-f(x)$ is of Baire class α for every $h \in \mathbb{R}$. Does it follow that f is of class α, too?*

R.Filipów and I.Recław in [1] showed that the Continuum Hypothesis (CH) implies the negative answer to Problem 2. For all *bounded* Borel functions, Problem 3 can be answered affirmatively using a theorem of Louveau about measurability of integral operations. See [5, Section 7], where Problem 3 appears as Problem 7.3.

In Section 2 of this article, Problem 3 is answered affirmatively for all *infinite* α (Theorem 1). Using the same idea, we show in Section 3 that a very strong form of affirmative answer to Problem 2 holds under the axiom of determinacy or in the Solovay model (Theorem 2). We do not insist that this solves Laczkovich's problem, since the full axiom of choice fails in these models. Another fragment of affirmative answer is that if the Lebesgue measure is ω_2-additive and if every projective set is measurable, then every projective f satisfies the conclusion of Problem 2 (Theorem 3).

1 Preliminaries

We need some notions and notations from Descriptive Set Theory. Chapters 11 and 25 of [2] are handy reference.

In this note, against set-theorists' custom, $\mathbb{R}$ refers to the real line. We give the set ω of non-negative integers the discrete topology and the infinite product ${}^{\omega}\omega$ the product topology. In our exposition, all spaces involved are of the form $\mathbb{R}^{\ell} \times \omega^m \times ({}^{\omega}\omega)^n$ (ℓ, m, n being non-negative integers). We fix a recursive enumeration $\langle I_i : i \in \omega \rangle$ of all open intervals with rational endpoints.

Let us denote by $\boldsymbol{B}$ the class of all Borel sets. $\boldsymbol{B}$ ramifies into the hierarchy: $\boldsymbol{B} = \bigcup_{1 \leq \alpha < \omega_1} \boldsymbol{\Sigma}^0_\alpha = \bigcup_{1 \leq \alpha < \omega_1} \boldsymbol{\Pi}^0_\alpha$ as defined in Chapter 11 of [2].

Let us denote by $\boldsymbol{P}$ the class of all projective sets. $\boldsymbol{P}$ also ramifies into the hierarchy: $\boldsymbol{P} = \bigcup_{n=1}^{\infty} \boldsymbol{\Sigma}^1_n = \bigcup_{n=1}^{\infty} \boldsymbol{\Pi}^1_n$. Here, $\boldsymbol{\Sigma}^1_1$ is the class of analytic sets (continuous images of Borel sets), $\boldsymbol{\Pi}^1_1$ is the class of coanalytic sets (i.e., complements of analytic sets), $\boldsymbol{\Sigma}^1_2$ is the class of continuous images of $\boldsymbol{\Pi}^1_1$ sets (so-called PCA sets), $\boldsymbol{\Pi}^1_2$ is the class of complements of $\boldsymbol{\Sigma}^1_2$ sets (so-called CPCA sets). We also define $\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$. Thus by the Suslin theorem $\boldsymbol{\Delta}^1_1 = \boldsymbol{B}$. Each $\boldsymbol{\Delta}^1_n$ forms a countably additive Boolean algebra, while none of $\boldsymbol{\Sigma}^1_n$ nor $\boldsymbol{\Pi}^1_n$ is closed under complements. The class $\boldsymbol{P}$ of all projective sets forms a finitely additive Boolean algebra, but not closed under countable unions.

In the proof of Theorem 1, we need the notions of *lightface* classes Σ^0_α, Δ^1_1, etc. See Chapter 25 of [2] and Section 1 of [6].

Each of non-selfdual classes from Borel and projective hierarchies (i.e., $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$) admits a *universal set.* Let Γ be one of those classes. A set

$E \subset {}^\omega\omega \times \mathbb{R}$ is a universal Γ set provided that $E \in \Gamma$ and that every $A \subset \mathbb{R}$ in Γ is a section of E (that is to say, there is a $c \in {}^\omega\omega$ such that $A = E_c = \{\, x \in \mathbb{R} : \langle c, x\rangle \in E \,\}$.) The classes which are closed under complements, such as $\boldsymbol{\Delta}^1_n$, $\boldsymbol{B}$ and $\boldsymbol{P}$, do not admit a universal set.

In the proof of Theorem 2, we also need the concept of a Π^1_1 *coding system* for Borel sets. That is a triple (W, B^+, B^-) such that

(i) $W \subset {}^\omega\omega$ and $B^+, B^- \subset {}^\omega\omega \times \mathbb{R}$;

(ii) W, B^+ and B^- are Π^1_1;

(iii) if $c \in W$, then $\forall x \in \mathbb{R}\,[\langle c, x\rangle \in B^+ \iff \langle c, x\rangle \notin B^-\,]$; and

(iv) for every Borel set $B \subset \mathbb{R}$ there is $c \in W$ such that $x \in B \iff \langle c, x\rangle \in B^+$.

Thus section of B^+ at every $c \in W$ is Borel and conversely every Borel subset of $\mathbb{R}$ is a section of B^+ at some $c \in W$. Such a coding system exists. See [2, page 504].

Let Γ be one of the classes in Borel and projective hierarchies. We say a function $f : \mathbb{R} \to \mathbb{R}$ is Γ-*measurable* if for every open interval I the preimage $f^{-1}[I]$ belongs to the class Γ. f is said to be Γ-*recursive* if set of all pairs $\langle x, i\rangle$ such that $f(x) \in I_i$ is, as a subset of $\mathbb{R} \times \omega$, belongs to Γ. While Γ-recursiveness is, in general, a much finer notion than Γ-measurability, two notions coincide when Γ is closed under arbitrary countable unions.

Every Borel function is $\boldsymbol{\Sigma}^0_\alpha$-measurable for some countable ordinal α. A function is of Baire class α if and only if it is $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable. Relying on this fact, we stop mentioning Baire classes of functions hereafter.

We say a function $f : \mathbb{R} \to \mathbb{R}$ to be *projective* if its graph is a projective subset of $\mathbb{R} \times \mathbb{R}$. This is equivalent to f being $\boldsymbol{P}$-recursive, slightly stronger than being $\boldsymbol{P}$-measurable, since $\boldsymbol{P}$ is not closed under countable unions. It is immediate from the definition that a functions is projective if and only if it is $\boldsymbol{\Sigma}^1_n$-measurable for some $n \in \omega$.

Note that since each of $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$ are closed under countable unions, and since we are dealing only with total (i.e., defined everywhere) functions, $\boldsymbol{\Sigma}^1_n$-recursive, $\boldsymbol{\Sigma}^1_n$-measurable, $\boldsymbol{\Pi}^1_n$-recursive, $\boldsymbol{\Pi}^1_n$-measurable, $\boldsymbol{\Delta}^1_n$-recursive and $\boldsymbol{\Delta}^1_n$-measurable are all the same thing. Therefore $\boldsymbol{\Sigma}^1_1$-measurable functions are precisely Borel functions. This is equivalent to the graph of function being analytic. On the other hand, functions with coanalytic graph are not necessarily $\boldsymbol{\Pi}^1_1$-measurable ($= \boldsymbol{B}$-measurable). In fact, it is consistent that there exists a Lebesgue non-measurable function with coanalytic graph.

We define the difference function as follows. Given $f : \mathbb{R} \to \mathbb{R}$ and $h \in \mathbb{R}$, the function $\triangle_h f : \mathbb{R} \to \mathbb{R}$ assigns $f(x+h) - f(x)$ to each x. Clearly, if f is continuous, Borel measurable, Lebesgue measurable, etc., so is $\triangle_h f$ for every $h \in \mathbb{R}$.

2 Borel functions with $\mathbf{\Sigma}^0_\alpha$ differences

Here we are going to prove

Theorem 1. *Let $\alpha > 0$ be a countable ordinal. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\triangle_h f$ is $\mathbf{\Sigma}^0_\alpha$-measurable for every $h \in \mathbb{R}$. Then f is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable.*

This partially answers Laczkovich's Problem 3. In particular, the problem is settled affirmatively for all *infinite* α since $\alpha \geq \omega$ implies $1 + \alpha = \alpha$.

First of all, note that we may assume $\alpha > 1$ because the other case of $\alpha = 1$ is covered by de Brujn's theorem.

Let $E_\alpha \subset {}^\omega\omega \times \mathbb{R}$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Then our assumption on f can be written

$$\forall h \in \mathbb{R} \forall i \in \omega \exists c \in {}^\omega\omega \forall x \in \mathbb{R}[\, E_\alpha(c, x) \iff \triangle_h f(x) \in I_i\,]. \tag{1}$$

Let $P(h, i, c)$ denote the subformula "$\forall x \in \mathbb{R}[\dots]$" in the above statement. Then P is, as a subset of $\mathbb{R} \times \omega \times {}^\omega\omega$, coanalytic. We have $\forall h \forall i \exists c P(h, i, c)$. Then by Kondô's uniformization theorem (see [2, Theorem 25.36] or [10, Theorem 5.14.1]), there exists a function $C : \mathbb{R} \times \omega \to {}^\omega\omega$ with coanalytic graph such that

$$\forall h \in \mathbb{R} \forall i \in \omega\, P(h, i, C(h, i)).$$

We want to approximate the selection function C by Borel measurable functions. But this is not possible in general, since a function with coanalytic graph may even fail to be Lebesgue measurable. Thus at this point, *we have to introduce the extra hypothesis that every real function with coanalytic graph is Lebesgue measurable.* This is equivalent to Lebesgue measurablity of all $\mathbf{\Delta}^1_2$ sets of reals. The hypothesis holds, for example, under Martin's axiom with the negation of CH. The important point is that we can always "force" it by a standard forcing machinery. At the end of proof, *this extra hypothesis will eventually be removed using the absoluteness argument.*

By virtue of the extra hypothesis, $C(h, i)$ is a measurable function of h for every $i \in \omega$. By Lusin's theorem, there are compact sets $K_n \subset \mathbb{R}$ ($n \in \omega$) such that $\mathbb{R} \setminus \bigcup_{n\in\omega} K_n$ is null and the restriction of the function C to $K_n \times \omega$ is continuous. For each n and i in ω, define functions $g_{n,i} : K_n \to {}^\omega\omega$ by $g_{n,i}(h) = C(h, i)$.

Let $A = \bigcup_{n\in\omega} K_n$. Then $A+A = \mathbb{R}$. So $\mathbb{R} = \bigcup_{n,m\in\omega}(K_n+K_m)$. If $h \in K_n+K_m$, then $K_n \cap (h-K_m) \neq \emptyset$. Therefore we can let $u_{n,m}(h) = \min(K_n \cap (h-K_m))$ and $v_{n,m}(h) = h - u_{n,m}(h)$. All these functions are $\mathbf{\Sigma}^0_2$-measurable since $u_{n,m}(h)$ (resp. $v_{n,m}(h)$) is upper (resp. lower) semi-continuous.

Now let $\{L_j\}_{j\in\omega}$ enumerate $\{K_n + K_m\}_{n,m\in\omega}$. Let u_j and v_j be corresponding functions $u_{n,m}$ and $v_{n,m}$. For each $h \in \mathbb{R}$ let $H_0(h) = u_j(h)$ and $H_1(h) = v_j(h)$ for the unique j such that $h \in L_j \setminus \bigcup_{j'<j} L_{j'}$. Then H_0 and H_1 are $\mathbf{\Sigma}^0_2$-measurable functions defined on $\mathbb{R}$. For every $h \in \mathbb{R}$ we have $H_0(h), H_1(h) \in A$ and $h = H_0(h) + H_1(h)$.

Let $i \in \omega$. Since $H_0(h) \in A$, we have $C(H_0(h), i) = g_{n,i}(H_0(h))$ for any $n \in \omega$ such that $H_0(h) \in K_n$. Therefore $\triangle_{H_0(h)} f(x) \in I_i$ if and only if $(\exists n)[\, H_0(h) \in K_n \wedge E_\alpha(g_{i,n}(H_0(h)), x)\,]$. Similar equivalence holds for $H_1(h)$. Since $h = H_0(h)+H_1(h)$, the equation $\triangle_h f(x) = \triangle_{H_0(h)} f(x+H_1(h)) + \triangle_{H_1(h)} f(x)$ holds. We thus have

$$\begin{aligned}
\triangle_h f(x) \in I_i \iff & \exists i_0 \exists i_1 [\, I_{i_0} + I_{i_1} \subset I_i \\
& \qquad \wedge \triangle_{H_0(h)} f(x + H_1(h)) \in I_{i_0} \\
& \qquad \wedge \triangle_{H_1(h)} f(x) \in I_{i_1}\,] \\
\iff & \exists i_0 \exists i_1 \exists n \exists m [\, I_{i_0} + I_{i_1} \subset I_i \\
& \qquad \wedge H_0(h) \in K_n \\
& \qquad \wedge H_1(h) \in K_m \\
& \qquad \wedge E_\alpha(g_{n,i_0}(H_0(h)), x + H_1(h)) \\
& \qquad \wedge E_\alpha(g_{m,i_1}(H_1(h)), x)\,]
\end{aligned}$$

for every $h \in \mathbb{R}$, $i \in \omega$ and $x \in \mathbb{R}$. This equivalence establishes a definition of $\triangle_h f(x)$ as functions of two variables h and x.

If you substitute $\mathbf{\Sigma}^0_2$-measurable functions $g_{n,i_0}(H_0(h))$ and $g_{m,i_1}(H_1(h))$ into $\mathbf{\Sigma}^0_\alpha$ formulas E_α, the results are $\mathbf{\Sigma}^0_{1+\alpha}$. Subformulas $H_0(h) \in K_n$ and $H_1(h) \in K_m$ are $\mathbf{\Pi}^0_2$, hence $\mathbf{\Sigma}^0_{1+\alpha}$ if $\alpha > 1$. So the last formula is $\mathbf{\Sigma}^0_{1+\alpha}$. From this it follows that f is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable.

This almost completes the proof of Theorem 1. But remember that we have introduced an extra hypothesis that every $\mathbf{\Delta}^1_2$ sets are measurable. Now we remove this hypothesis.

Lemma 2.1. *Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a (lightface) Δ^1_1-recursive function and ξ be a recursive ordinal. Then the set of x such that the sectional function $F_x(y) = F(x, y)$ is $\mathbf{\Sigma}^0_\xi$-measurable is Π^1_1.*

PROOF: See [6, §3, Lemma 4]. Apply that result with $A = \{\, \langle x, \langle y, i\rangle\rangle \in \mathbb{R} \times (\mathbb{R} \times \omega) : F(x, y) \in I_i \,\}$ and $B = \mathbb{R} \times (\mathbb{R} \times \omega) \setminus A$. ■

Let r be a real such that f is $\Delta^1_1(r)$-recursive and $\alpha < \omega^r_1$. That is to say, α is the order-type of a wellordering relation on ω which is recursive in r. Then the relativized version of Lemma 2.1 implies that our assumption (1) is a $\Pi^1_1(r)$ statement. Therefore its truth persists in every transitive model of set theory which contains the parameter r among its members.

Let $\mathbb{P}$ be a notion of forcing that forces measurability of $\mathbf{\Delta}^1_2$ sets (measure algebra on ${}^{\omega_1}2$ should work fine.) In a $\mathbb{P}$-generic extension of the universe, the condition (1) still holds. Therefore f is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable as we have just proved.

Still in the "extended" universe, apply the Louveau Separation Theorem ([6, Theorem A], see also [7, Chapter 8] and [8, Section 6 of IV]) to the set $\{\, \langle x, i\rangle \in \mathbb{R} \times \omega : f(x) \in I_i \,\}$. We obtain a real s such that $s \in \Delta^1_1(r)$ and f is $\Sigma^0_{1+\alpha}(s)$-recursive. Being Δ^1_1 definable from r the real s belongs to the "original" universe. Therefore by absoluteness of $\mathbf{\Pi}^1_1$ formulas again, f is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable in the original universe. ■

Thus we solved Laczkovich's Problem 3 for infinite α. We should admit this is not a quite satisfactory result since (a) it leaves the cases for finite α unsettled, and (b) the argument involves metamathematical tools such as generic extensions, absoluteness, lightface classes, etc. Anyway, this result and Laczkovich's solution for bounded functions strongly suggest that Problem 3 would be solved affirmatively for all α.

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3 Measure Uniformization Principle

The Measure Uniformization Principle (MUP) is the following statement: *Let $X \subset \mathbb{R} \times \mathbb{R}$ and suppose that its x-section $X_x = \{ y : \langle x, y \rangle \in X \}$ is nonempty for almost every $x \in \mathbb{R}$. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) \in X_x$ holds for almost every x* ('almost every' refers to Lebesgue measure here). This statement was proposed first by J.Mycielski who pointed out that Solovay's model of Lebesgue measurability ([9]) satisfies it. Then Solovay observed that the Axiom of Determinacy implies MUP.

It is clear that MUP is incompatible with the full Axiom of Choice (AC). When we talk about consequences of MUP, we thus have to use a weakend version of AC, such as the Principle of Dependent Choice (DC): *every partial ordering without a maximal element admits an infinite ascending chain.*

Using the main idea of Theorem 1, we obtain:

Theorem 2. *(in ZF+DC) Assume MUP. Let $f : \mathbb{R} \to \mathbb{R}$. If $\triangle_h f$ is Borel for almost every h, then f is Borel.*

Proof: Let (W, B^+, B^-) be a Π^1_1 coding system of Borel subsets of $\mathbb{R}$. Define $P(h, i, c)$ by

$$\begin{aligned} P(h,i,c) \iff & c \in W \\ & \wedge \forall x \in \mathbb{R}[\, \triangle_h f(x) \in I_i \iff \langle c, x\rangle \in B^+ \iff \langle c, x \rangle \notin B^- \,] \end{aligned}$$

Then by our assumption on f, for almost every h we have $\forall i \in \omega \exists c\, P(h,i,c)$. Therefore by MUP there is a Borel function $C : \mathbb{R} \times {}^\omega\omega \to {}^\omega\omega$ such that for almost every $h \in \mathbb{R}$ we have $\forall i \in \omega P(h, i, C(h,i))$. As in Theorem 1, there are a $\mathbf{\Sigma}^0_2$ set $A \subset \mathbb{R}$ and $\mathbf{\Sigma}^0_2$-measurable functions H_0 and H_1 such that $\forall h \in A \forall i \in \omega P(h, i, C(h, i))$, $H_0(h) \in A$, $H_1(h) \in A$ and $h = H_0(h) + H_1(h)$

for every $h \in \mathbb{R}$. Then we have for every $h \in \mathbb{R}$, $i \in \omega$ and $x \in \mathbb{R}$,

$$\begin{aligned}\triangle_h f(x) \in I_i \iff & \exists i_0 \exists i_1 [\, I_{i_0} + I_{i_1} \subset I_i \\ & \qquad \wedge \langle C(H_0(h), i_0), x + H_1(h)\rangle \in B^+ \\ & \qquad \wedge \langle C(H_1(h), i_1), x\rangle \in B^+ \,] \\ \iff & \exists i_0 \exists i_1 [\, I_{i_0} + I_{i_1} \subset I_i \\ & \qquad \wedge \langle C(H_0(h), i_0), x + H_1(h)\rangle \notin B^- \\ & \qquad \wedge \langle C(H_1(h), i_1), x\rangle \notin B^- \,].\end{aligned}$$

This defines $\triangle_h f(x)$ as $\mathbf{\Delta}^1_1$-recursive function of two variables h and x. Hence f is Borel measurable. ■

Now, let $f : \mathbb{R} \to \mathbb{R}$ be projective. Let G_α be the set of all $h \in \mathbb{R}$ such that $\triangle_h f$ is $\mathbf{\Sigma}^0_\alpha$-measurable. Then $\langle G_\alpha : 1 \le \alpha < \omega_1\rangle$ forms an increasing ω_1-chain of subgroups of $(\mathbb{R}, +)$.

Lemma 3.1. *Each G_α is a projective subset of $\mathbb{R}$. More specifically, if f is $\mathbf{\Sigma}^1_n$-measurable, then G_α is $\mathbf{\Sigma}^1_{n+1}$ for each α.*

Proof: Let $E_\alpha \subset {}^\omega\omega \times \mathbb{R}$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Suppose that f is $\mathbf{\Sigma}^1_n$-measurable. It follows that f is in fact $\mathbf{\Delta}^1_n$-measurable. Then $\triangle_h f(x)$ is also $\mathbf{\Delta}^1_n$-measurable as a function of two variables h and x. We thus obtain a $\mathbf{\Sigma}^1_{n+1}$-definition of G_α as follows:

$$\forall i \in \omega \exists c \in {}^\omega\omega \forall x \in \mathbb{R}[\, \triangle_h f(x) \in I_i \iff E_\alpha(c, x)\,]. \qquad ■$$

Theorem 3. *Suppose that the Lebesgue measure is ω_2-additive and that every projective set is measurable. Let $f : \mathbb{R} \to \mathbb{R}$ be a projective function such that $\triangle_h f$ is Borel for every $h \in \mathbb{R}$. Then there is a countable ordinal α such that $\triangle_h f$ is $\mathbf{\Sigma}^0_\alpha$-measurable for all $h \in \mathbb{R}$.*

Proof: By the assumption on f, we have $\mathbb{R} = \bigcup_{1 \le \alpha < \omega_1} G_\alpha$. The whole real line is covered by ω_1 projective sets, which are by the assumption Lebesgue measurable. Since the Lebesgue measure is ω_2-additive, some G_α must be of positive measure. But being a measurable subgroup of $(\mathbb{R}, +)$ such G_α must be the whole line. This means $\triangle_h f$ is $\mathbf{\Sigma}^0_\alpha$-measurable for every $h \in \mathbb{R}$. ■

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