Lebesgue’s Basis Theorem

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A real number \( x \) is said to be normal to base 10 if the decimal expansion

\[
x = [x].d_1d_2d_3\ldots \quad (d_i = 0,\ldots, 9)
\]

contains every finite string of digits at the equal frequency (depending only on the length of that string.) Such a number, as an array of digits, behave like a random-number table. Of course, any true random number (whatever it is) should not only be normal to base 10 but also be normal to any other bases in the obvious sense. A real number which is normal to all bases \( b = 2,3,\ldots \) is called an absolutely normal number.

E. Borel proved that almost all real numbers are absolutely normal. However, he could not give any example of absolutely normal numbers. As for the classical irrational numbers such as \( \pi, e, \sqrt{2} \) etc., it is not yet known whether any of them is absolutely normal (nor even normal to base 10.) Borel considered it paradoxical that one could give the concept of an absolutely normal number prove the existence of such a number without knowing any example.

H. Lebesgue, asking whether one could prove the existence of a mathematical being without defining it, wrote: if the concept is not illusory, a proof of the existence should admit precision which leads to the definition of one of such thing. He asserted that one could extract a definition of an absolutely normal number from Borel’s argument.

In modern terminology, that Lebesgue has demonstrated can be formulated as follows: Let \( K \) be a \( \Pi^0_1 \) subset of the unit closed interval. Let \( s \) be the Lebesgue measure of \( K \). If \( s > 0 \) then \( K \) contains a member which are recursive in \( s \). Of course, as Lebesgue did not know modern recursion theory, this differs from what he has really said. However, I would like to call the result “Lebesgue’s basis theorem.”

Sierpiński and Lebesgue independently gave direct existence proof of absolutely normal numbers. Their pioneering works were followed by lots of results which lead to explicit construction procedures of normal numbers. See Section 8, Chapter 1 of [1].

1 Lebesgue’s existence proof of normal numbers

What Lebesgue has proved in his article [2] is, in modern terminology, the following
Theorem 1 Let $K$ be a $\Pi^0_1$ subset of the unit closed interval. Let $r = m(K)$ be the Lebesgue measure of $K$. If $r > 0$ then $K$ has members which are recursive in $r$.

In [2], Lebesgue has concluded that one can explicitly define an absolutely normal number using this argument.

By the theorem, a $\Pi^0_1$ set whose Lebesgue measure is a positive recursive real contains an infinitely many recursive members. Either condition here on the measure, positiveness or recursiveness, cannot be dropped. There exist a $\Pi^0_1$ set (of measure zero) without a recursive member (Kleene) and $\Pi^0_1$ set of (non-recursive) positive measure without a recursive member (Tanaka). See [3].

H. Lebesgue, who lived the years before modern recursion theory, did not state the result in this form. The result stated here has been extracted by the present author from Lebesgue’s argument. However, we would like to call it Lebesgue’s basis theorem because H. Lebesgue was the first person who noticed the importance of the basis problem. In other words, Lebesgue was the first person who claimed that mathematicians should take some care of effectiveness of a mathematical object when they introduce it.

Let us get into the proof of Theorem 1. Let $K$ be a $\Pi^0_1$ subset of the unit interval $[0,1]$. Suppose $r = m(K) > 0$. We are required to give a way to find a member of $K$ which is recursive in $r$.

From now on we assume $r$ to be recursive. The general case would be obtained by straightforward relativization procedure.

Lemma 1.1 There is a recursive sequence $\{I_n\}$ of rational open intervals such that

$$[0,1] \setminus K \subset \bigcup_{n=1}^\infty I_n$$

and that the sum of the lengths

$$s^* = \sum_{n=1}^\infty |I_n|$$

is a recursive real less than 1.

Proof: Fix a small positive rational number $\varepsilon$. By the assumption that $K$ is $\Pi^0_1$ there exists a recursive sequence $\{J_n\}$ of rational open intervals such that

$$K = [0,1] \setminus \bigcup_{n=1}^\infty J_n.$$

From the assumption on the measure of $K$, the real

$$s = m\left([0,1] \cap \bigcup_{n=1}^\infty J_n\right) = 1 - m(K)$$

is a recursive real less than 1.
is a recursive real less than 1. Therefore we can find a recursive sequence of increasing integers
\[ N_1 < N_2 < \cdots < N_k < \cdots \]
such that \( m(J_1 \cup \cdots \cup J_{N_k}) > (1 - 2^{-(k+1)}\varepsilon)s \). Put \( E_1 = (J_1 \cup J_{N_1}) \) and \( E_{k+1} = (J_{N_{k+1}} \cup \cdots \cup J_{N_k+1}) \setminus (J_1 \cup \cdots \cup J_{N_k}) \). Each \( E_k \) is union of finitely many rational half-open intervals. The measure of \( E_k \) satisfies
\[ m(E_k) < 2^{-(k+1)}\varepsilon s. \]

One can easily find, for each \( k \), a set \( H_k \) which is the open of finite family of pairwise disjoint rational open intervals satisfying
\[ E_k \subset H_k \quad \text{and} \quad m(H_k) < 2^{-(k+1)}\varepsilon s. \]

Moreover, such \( H_k \) and the open intervals that it consists of can be found in a uniform manner recursive in \( k \).

Now let \( \{I_n\} \) be a recursive enumeration of all rational open intervals which constitute \( H_k \)'s. As for their lengths, we have
\[ s^* = \sum_{n=1}^{\infty} |I_n| = \sum_{k=1}^{\infty} m(H_k) \]
with \( \{m(H_k)\} \) being a recursive sequence of rational numbers tending to zero with \( 2^{-k} \). Therefore its sum \( s^* \) is a recursive real. By the choice of \( \{N_k\} \) we have \( s^* < (1 + \varepsilon)s \). Therefore \( s^* \) is smaller than 1 if \( \varepsilon \) is small enough. (QED)

**Lemma 1.2** Let \( \{I_n\} \) be as in Lemma 1.1. Then the complement \([0, 1] \setminus \bigcup_{n=1}^{\infty} I_n\) contains a recursive real.

**Proof:** To each finite binary sequence \( \sigma \in \{0, 1\}^{<\infty} \), we assign closed interval \( B_\sigma \) as follows: \( B_\emptyset = [0, 1] \); \( B_0 \) is its left half \([0, 1/2] \); \( B_1 \) is the right half \([1/2, 1] \); having defined \( B_\sigma \), the next generation \( B_\sigma \setminus 0 \) and \( B_\sigma \setminus 1 \) are the left and right halves of \( B_\sigma \) respectively.

Suppose that we are given a recursive sequence \( \varepsilon_k \setminus 0 \) of positive rational numbers tending to zero. We will describe later how fast the sequence decreases. Since the sum \( s^* \) of lengths of \( I_n \) is a recursive real, we can find a recursive sequences increasing integers \( N_k \) such that
\[ \sum_{n > N_k} |I_n| < \varepsilon_k \quad (k = 1, 2, \ldots). \]

Now, since \( \sum_{n=1}^{N_k} |I_n| \) is smaller than \( s^* \) and
\[ \sum_{n=1}^{N_1} |I_n| = \sum_{n=1}^{N_1} |I_n \cap B_0| \sum_{n=1}^{N_1} |I_n \cap B_1|, \]
Either $\sum_{n=1}^{N_1} |I_n \cap B_0|$ or $\sum_{n=1}^{N_1} |I_n \cap B_1|$ is smaller than $s^*/2$. Let $\alpha(0) = 0$ if so is the former, otherwise let $\alpha(0) = 1$. Then anyway we have

$$\sum_{n=1}^{N_1} |I_n \cap B_{\alpha(0)}| < \frac{s^*}{2}.$$  

From this it follows that

$$\sum_{n=1}^{N_2} |I_n \cap B_{\alpha(0)}| + \sum_{n=N_1+1}^{N_2} < \frac{s^*}{2} + \varepsilon_1.$$  

Arguing just as before, choose $\alpha(1) \in \{0, 1\}$ so that

$$\sum_{n=1}^{N_2} |I_n \cap B_{\alpha(0)\alpha(1)}| < \frac{s^*}{2} + \varepsilon_1.$$  

Continuing inductively, choose $\alpha(k) \in \{0, 1\}$ so that

$$\sum_{n=1}^{N_k} |I_n \cap B_{\alpha(0)\cdots\alpha(k-1)}| < \frac{s^*}{2^k} + \varepsilon_1 + \cdots + \varepsilon_{k-1}.$$  

Thus if the sequence $\{\varepsilon_k\}$ has been chosen so that

$$2\varepsilon_1 + 2^2 \varepsilon_2 + \cdots + 2^k \varepsilon_k + \cdots \leq 1 - s^*,$$

then we have for each $k$ that

$$\sum_{n=1}^{N_k} |I_n \cap B_{\alpha(0)\cdots\alpha(k-1)}| < \frac{1}{2^k} = |B_{\alpha(0)\cdots\alpha(k-1)}|.$$  

In order to make this, it is sufficient that $\varepsilon_k$ satisfies

$$\varepsilon_k < \frac{1 - s^*}{2 \cdot 3^k} \quad (k = 1, 2, \ldots).$$  

Such recursive sequence $\{\varepsilon_k\}$ can be found since $s^*$ is a recursive real.

The binary sequence $\alpha \in \{0, 1\}$ thus obtained is recursive. Therefore the limiting point $\xi$ of the decreasing sequence of intervals:

$$\{\xi\} = \bigcap_{k=1}^{\infty} B_{\alpha(0)\cdots\alpha(n-1)}$$

is a recursive real. That $\xi$ does not belong to the union of $\{I_n\}$ is verified as follows: Fix an arbitrary integer $n$. If $k$ is so large enough that $N_k \geq n$, then $B_{\alpha(0)\cdots\alpha(k-1)} \not\subseteq I_n$ because

$$|I_n \cap B_{\alpha(0)\cdots\alpha(k-1)}| < |B_{\alpha(0)\cdots\alpha(k-1)}|.$$  

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Therefore no neighborhood of \( \xi \) is contained in \( I_n \). But, being an open interval, \( I_n \) would be an neighborhood of \( \xi \) if \( \xi \in I_n \). Therefore \( \xi \) does not belong to \( I_n \). For \( n \) being arbitrary this proves \( \xi \notin \bigcup_{n=1}^{\infty} I_n \). Thus the lemma is proved. (QED)

Theorem 1 follows immediately from these two lemmas. Here we have shown how to define a member of large compact subset of \([0, 1]\), using the measure of the set as a parameter. In mathematical practice, the measure of an explicitly given compact set are expected to be explicitly computable. Therefore Theorem 1 supports our intuition that when a large compact set is given, we can choose an element of it quite effectively. Although it is not always the case that a \( \Pi^0_1 \) set of positive measure has a recursive member, a counterexample is hardly found in mathematical practice.

References

