Lebesgue's Basis Theorem

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A real number x is said to be *normal to base 10* if the decimal expansion

$$x = [x].d_1 d_2 d_3 \dots \qquad (d_i = 0, \dots, 9)$$

contains every finite string of digits at the equal frequency (depending only on the length of that string.) Such a number, as an array of digits, behave like a random-number table. Of course, any true randome number (whatever it is) should not only be normal to base 10 but also be normal to any other bases in the obvious sense. A real number which is normal to all bases $b = 2, 3, \dots$ is called an absolutely normal number.

E. Borel provedthat almostall real numbersare absolutely normal. However, he could not give any example of absolutely normal numbers. As for the classical irrational numbers such as π, e, $\sqrt{2}$ etc., it is not yet known whether any of them is absolutely normal (nor even normal to base 10.) Borel considered it paradoxical that one could give the concept of an absolutely normal number prove the existence of such a number without knowing any example.

H. Lebesgue, asking whether one couldprove the existence of a mathematical being without defining it, wrote: if the concept is not illusory, a proof of the existence should admit precision which leads to the definition of one of such thing. He asserted that one could extract a definition of an absolutely normal number from Borel's argument.

In modern terminology, that Lebesgue has demonstrated can be formulated as follows: Let K be a Π^0_1 subset of the unit closed interval. Let s be the Lebesgue measure of K. If $s > 0$ then K contains a member which are recursive in s. Of course, as Lebesgue did not know modern recursion theory, this differs from what he has really said. However, I wouldlike to call the result "Lebesgue's basis theorem."

Sierpinski and Lebesgue independently gave direct existence proof of absolutely normal numbers. Their pioneering works were followed by lots of results which leaded to explicit construction procedures of normal numbers. See Section 8, Chapter 1 of [1].

1 Lebesgue's existence proof of normal numbers

What Lebesgue has proved in his article [2] is, in modern terminology, the following

Theorem 1 *Let K be a Π^0_1 subset of the unit closed interval. Let $r = m(K)$ be the Lebesgue measure of K. If $r > 0$ then K has members which are recursive in r.*

In [2], Lebesgue has concluded that one can explicitly define an absolutely normal number using this argument.

By the theorem, a Π^0_1 set whose Lebesgue measure is a positive recursive real contains an infinitely many recursive members. Either condition here on the measure, positiveness or recursiveness, cannot be dropped. There exist a Π^0_1 set (of measure zero) without a recursive member (Kleene) and Π^0_1 set of (non-recursive) positive measure without a recursive member (Tanaka). See [3].

H. Lebesgue, who lived the years before modern recursion theory, did not state the result in this form. The result stated here has been extracted by the present author from Lebesgue's argument. However, we would like to call it *Lebesgue's basis theorem* because H. Lebesgue was the first person who noticed the importance of the basis problem. In other words, Lebesgue was the first person who claimed that mathematicians should take some care of effectiveness of a mathematical object when they introduce it.

Let us get into the proof of Theorem 1. Let K be a Π^0_1 subset of the unit interval $[0, 1]$. Suppose $r = m(K) > 0$. We are required to give a way to find a member of K which is recursive in r.

From now on we assume r to be recursive. The general case would be obtained by straightforward relativization procedure.

Lemma 1.1 *There is a recursive sequence $\{I_n\}$ of rational open intervals such that*

$$[0, 1] \setminus K \subset \bigcup_{n=1}^{\infty} I_n$$

and that the sum of the lengths

$$s^* = \sum_{n=1}^{\infty} |I_n|$$

is a recursive real less than 1*.*

PROOF: Fix a small positive rational number ε. By the assumption that K is Π^0_1 there exists a recursive sequence $\{J_n\}$ of rational open intervals such that

$$K = [0, 1] \setminus \bigcup_{n=1}^{\infty} J_n.$$

From the assumption on the measure of K, the real

$$s = m\left([0, 1] \cap \bigcup_{n=1}^{\infty} J_n\right) = 1 - m(K)$$

is a recursive real less than 1. Therefore we can find a recursive sequence of increasing integers

$$N_1 < N_2 < \cdots < N_k < \cdots$$

such that $m(J_1 \cup \cdots \cup J_{N_k}) > (1 - 2^{-(k+1)}\varepsilon)s$. Put $E_1 = (J_1 \cup J_{N_1})$ and $E_{k+1} = (J_{N_k+1} \cup \cdots \cup J_{N_{k+1}}) \setminus (J_1 \cup \cdots \cup J_{N_k})$. Each E_k is union of finitely many rational half-open intervals. The measure of E_k satifies

$$m(E_k) < 2^{-(k+1)}\varepsilon s.$$

One can easily find, for each k, a set H_k which is the open of finite family of pairwise disjoint rational open intervals satisfying

$$E_k \subset H_k \text{ and } m(H_k) < 2^{-(k+1)}\varepsilon s.$$

Moreover, such H_k and the open intervals that it consists of can be found in a uniform manner recursive in k.

Now let $\{I_n\}$ be a recursive enumeration of all rational open intervals which constitute H_k's. As for their lengths, we have

$$s^* = \sum_{n=1}^{\infty} |I_n| = \sum_{k=1}^{\infty} m(H_k)$$

with $\{m(H_k)\}$ being a recursive sequence of rational numbers tending to zero with 2^{-k}. Therefore its sum s^* is a recursive real. By the choice of $\{N_k\}$ we have $s^* < (1+\varepsilon)s$. Therefore s^* is smaller than 1 if ε is small enough. (QED)

Lemma 1.2 *Let $\{I_n\}$ be as in Lemma 1.1. Then the complement $[0,1] \setminus \bigcup_{n=1}^{\infty} I_n$ contains a recursive real.*

Proof: To each finite binary sequence $\sigma \in \{0,1\}^{<\infty}$, we assign closed interval B_σ as follows: $B_\emptyset$ is $[0,1]$; B_0 is its left half $[0,1/2]$; B_1 is the right half $[1/2,1]$; having defined B_σ, the next generation $B_{\sigma\frown 0}$ and $B_{\sigma\frown 1}$ are the left and right halves of B_σ respectively.

Suppose that we are given a recursive sequence $\varepsilon_k \searrow 0$ of positive rational numbers tending to zero. We will describe later how fast the sequence decreases. Since the sum s^* of lengths of I_n is a recursive real, we can find a recursive sequences increasing integers $N_k \nearrow \infty$ such that

$$\sum_{n>N_k} |I_n| < \varepsilon_k \quad (k = 1, 2, \dots).$$

Now, since $\sum_{n=1}^{N_1} |I_n|$ is smaller than s^* and

$$\sum_{n=1}^{N_1} |I_n| = \sum_{n=1}^{N_1} |I_n \cap B_0| \sum_{n=1}^{N_1} |I_n \cap B_1|,$$

Either $E\sum_{n=1}^{N_1}|I_n \cap B_0|$ or $\sum_{n=1}^{N_1}|I_n \cap B_1|$ is smaller than $s^*/2$. Let $\alpha(0) = 0$ if so is the former, otherwise let $\alpha(0) = 1$. Then anyway we have

$$\sum_{n=1}^{N_1}|I_n \cap B_{\alpha(0)}| < \frac{s^*}{2}.$$

From this it follows that

$$\sum_{n=1}^{N_2}|I_n \cap B_{\alpha(0)}| + \sum_{n=N_1+1}^{N_2} < \frac{s^*}{2} + \varepsilon_1.$$

Arguing just as before, choose $\alpha(1) \in \{0, 1\}$ so that

$$\sum_{n=1}^{N_2}|I_n \cap B_{\alpha(0)\alpha(1)}| < \frac{s^*}{2}^2 + \frac{\varepsilon_1}{2}.$$

Continuing inductively, choose $\alpha(k) \in \{0, 1\}$ so that

$$\sum_{n=1}^{N_k}|I_n \cap B_{\alpha(0)\alpha(1)\cdots\alpha(k-1)}| < \frac{s^*}{2^k} + \frac{\varepsilon_1}{2^{k-1}} + \cdots + \frac{\varepsilon_{k-1}}{2}.$$

Thus if the sequence $\{\varepsilon_k\}$ has been chosen so that

$$2\varepsilon_1 + 2^2\varepsilon_2 + \cdots + 2^k\varepsilon_k + \cdots \le 1 - s^*,$$

then we have for each k that

$$\sum_{n=1}^{N_k}|I_n \cap B_{\alpha(0)\cdots\alpha(k-1)}| < \frac{1}{2^k} = |B_{\alpha(0)\cdots\alpha(k-1)}|.$$

In order to make this, it is sufficient that ε_k satisfies

$$\varepsilon_k \le \frac{1 - s^*}{2 \cdot 3^k}, \quad (k = 1, 2, \dots).$$

Such recursive sequence $\{\varepsilon_k\}$ can be found since s^* is a recursive real.

The binary sequence $\alpha \in \{0, 1\}$ thus obtained is recursive. Therefore the limiting point ξ of the decreasing sequence of intervals:

$$\{\xi\} = \bigcap_{k=1}^{\infty} B_{\alpha(0)\cdots\alpha(n-1)}$$

is a recursive real. That ξ does not belong to the union of $\{I_n\}$ is verified as follows: Fix an arbitrary integer n. If k is so large enough that $N_k \ge n$, then $B_{\alpha(0)\cdots\alpha(k-1)} \not\subset I_n$ because

$$|I_n \cap B_{\alpha(0)\cdots\alpha(k-1)}| < |B_{\alpha(0)\cdots\alpha(k-1)}|.$$

Therefore no neighborhood of ξ is contained in I_n. But, being an open interval, I_n would be an neighborhood of ξ if $\xi \in I_n$. Therefore ξ does not belong to I_n. For n being arbitrary this proves $\xi \notin \bigcup_{n=1}^{\infty} I_n$. Thus the lemma is proved. (QED)

Theorem 1 follows immediately from these two lemmas. Here we have shown how to define a memberof large compactsubsetof $[0,1]$, using the measure of the set as a parameter. In mathematical practice, the measure of an explicitly given compact set are expected to be explicitly computable. Therefore Theorem 1 supports our intuition that when a large compact set is given, we can choose an element of it quite effectively. Although it is not always the case that a Π^0_1 set of positive measure has a recursive member, a counterexample is hardly found in mathematical practice.

References

[1] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. John Wiley & Sons, 1974.

[2] H. Lebesgue. Sur certaines démonstrations d'existence. *Bull. Soc. Math. France*, 45:132–144, 1917.

[3] Hisao Tanaka. On a Π^0_1 set of positive measure. *Nagoya Mathematical Journal*, 38:139–144, 1970.