A weak basis theorem for Π^1_2 sets of positive measure

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Abstract

We give a weak basis result for Π^1_2 sets of positive measure, which is closely related to our previous paper [2] in which we have assumed the existence of $0^\sharp$.

This note is devoted to the following

Theorem 1 *Let $s \in 2^\omega$ be a real such that $\aleph_1^L$ is a recursive-in-s ordinal. Then every Π^1_2 set of positive measure contains a $\Delta^1_1(s)$ member.*

This theorem is closely related to the main theorem of our previous paper [2]: *if $0^\sharp$ exists, then every Π^1_2 set of positive measure contains a member which is arithmetical in $0^\sharp$.* Indeed, letting $s = 0^\sharp$ the hypothesis of our present theorem is achieved and this almost (but not literally) proves our older theorem. The hypothesis in the present result is weaker than that of the "$0^\sharp$ version." Therefore, it seems to be applicable to wider context — See Section 3 for some discussion on L-generic models in which there is a Π^1_2 singleton s satisfying the hypothesis of Theorem 1.

1 Tools

Let us fix, once for all, a recursive bijection between $\omega \times \omega$ and ω. By the notation $\langle i,j \rangle$ we mean both the ordered pair and the integer which is assigned to this ordered pair by the fixed bijection. Each real $r \in 2^\omega$ codes a binary relation $\leq_r$ defined as

$$i \leq_r j \iff r(\langle i,j \rangle) = 1$$

Let **WO** be the set of reals $r \in 2^\omega$ such that $\leq_r$ well-orders ω. For $r \in \mathbf{WO}$, let $\|r\|$ be the order-type of the wellordering $\leq_r$. A countable ordinal ξ is said to be recursive-in-a if $\xi = \|r\|$ for some real $r \in \mathbf{WO}$ which is recursive in a.

The smallest ordinal which is not recursive-in-a is denoted by ω_1^a. Then ω_1^a equals the smallest ordinal $\xi > \omega$ such that the structure $(L_\xi(a), \in, a)$ is admissible. A real x is hyperarithmetical in a if and only if it is $\Delta^1_1(a)$ if and only if it belongs to $L_{\omega_1^a}(a)$.

For a countable ordinal ξ let $\mathbf{WO}(\xi)$ be the set of $r \in \mathbf{WO}$ with $\|r\| < \xi$. For each countable ξ, the set $\mathbf{WO}(\xi)$ is Borel. Indeed we have:

Lemma 1.1 *Let $s \in 2^\omega$. Let ξ be a recursive-in-s ordinal. Then $\mathbf{WO}(\xi)$ is a $\Delta^1_1(s)$ set.*

Proof. Let $r \in \mathbf{WO}$ be a real which is recursive in s and satisfies $\xi = \|r\|$. Then a real x belongs to $\mathbf{WO}(\xi)$ if and only if there is an order-preserving mapping of $(\omega, \leq_x)$ into an initial segment of $(\omega, \leq_r)$, if and only if $x \in \mathbf{WO}$ and there is no order-preserving mapping of $(\omega, \leq_r)$ into $(\omega, \leq_x)$. This gives a $\Delta^1_1(r)$ characterization of $\mathbf{WO}(\xi)$. $\square$

Let $s \in 2^\omega$ be a real such that $\aleph_1^L$ is a recursive-in-s ordinal. This readily implies $\aleph_1^L$ is countable. Under this assumption, every Π^1_2 set of reals is Lebesgue measurable. The main theorem is proved by examining how this measurability is realized in a certain effective way. To this end, we need two $\Delta^1_1(s)$ sets: Lemma 1.1 implies that the set $\mathbf{WO}(\aleph_1^L)$ of codes of constructibly countable well-ordering is $\Delta^1_1(s)$. Next we see that there is a $\Delta^1_1(s)$ set C of measure one consisting of random reals over L.

For a real $t \in 2^\omega$ and an integer $n \in \omega$, let $(t)_n$ be the real defined by: $(t)_n(i) = t(\langle n,i \rangle)$. Each real codes a countable sequence of reals in this way.

Lemma 1.2 *There is a $\Delta^1_1(s)$ real t such that*

$$\{ (t)_n : n \in \omega \} = 2^\omega \cap L.$$

Proof. For $2^\omega \cap L = 2^\omega \cap L_{\aleph_1^L}$, this set belongs to $L_{\omega_1^s}[s]$, the smallest admissible set containing s. Since $L_{\omega_1^s}[s]$ models "every set is countable," there exists in it a surjection $f : \omega \twoheadrightarrow 2^\omega \cap L_{\aleph_1^L}$. Let $t(\langle n,i \rangle) = f(n)(i)$. $\square$

Let $U \subset 2^\omega \times 2^\omega$ be a Π^0_2 set which is universal for $\mathbf{\Pi}^0_2$. Let t be a real as in Lemma 1.2. Let $C \subset 2^\omega$ be the following set

$$\begin{aligned} C = &\{ x \in 2^\omega : (\forall y \in 2^\omega \cap L)[\, \mu(U_y) = 0 \implies x \notin U_y \,] \} \\ = &\{ x \in 2^\omega : (\forall n)[\, \mu(U_{(t)_n}) = 0 \implies x \notin U_{(t)_n} \,] \}. \end{aligned}$$

where μ denotes the Lebesgue measure. Then C is a $\Delta^1_1(s)$ set suchthat $\mu(C) = 1$.

Lemma 1.3 *Every $x \in C$ is random over L. Consequently the equality $\aleph_1^{L[x]} = \aleph_1^L$ holds for all $x \in C$.* $\square$

2 Reducing Π^1_2 sets to $\Pi^1_1(s)$

Let P be a Σ^1_2 set of reals, then there is a recursive function $f : 2^\omega \times 2^\omega \to 2^\omega$ such that

$$x \in P \iff (\exists y)[\, f(x,y) \in \mathbf{WO} \,].$$

By the Shoenfield Absoluteness Lemma, it is equivalent to say

$$x \in P \iff (\exists y \in 2^\omega \cap L[x])[\, f(x,y) \in \mathbf{WO}\,].$$

In such a case, we have $f(x,y) \in L[x]$. So $\| f(x,y) \| < \aleph_1^{L[x]}$. It follows that

$$x \in P \iff (\exists y \in 2^\omega \cap L[x])[\, f(x,y) \in \mathbf{WO}(\aleph_1^{L[x]})\,].$$

By these observations, we have:

Lemma 2.1 *Let P be a Σ^1_2 set of reals, then there is a recursive function $f : 2^\omega \times 2^\omega \to 2^\omega$ such that*

$$x \in P \iff (\exists y)[\, f(x,y) \in \mathbf{WO}(\aleph_1^{L[x]})\,].$$

Now let A be a Π^1_2 set of reals. Put $P = 2^\omega \setminus A$, then by Lemmas 1.3 and 2.1, there is a recursive fucntion $f : 2^\omega \times 2^\omega \to 2^\omega$ such that

$$x \in C \implies [\, x \in A \iff (\forall y)[\, f(x,y) \notin \mathbf{WO}(\aleph_1^L)]\,].$$

Therefore we have

Lemma 2.2 *Let A and f as above. Then*

$$A \cap C = \{\, x \in 2^\omega : x \in C \;\&\; (\forall y)[\, f(x,y) \notin \mathbf{WO}(\aleph_1^L)] \,\}.$$

Consequently, $A \cap C$ is a $\Pi^1_1(s)$ set.

If A has positive Lebesgue measure, so is $A \cap C$, for C contains almost all reals. Being a $\Pi^1_1(s)$ set of positive measure, $A \cap C$ contains a $\Delta^1_1(s)$ real by the Sacks-Tanaka Basis Theorem ([4], Chap.IV, 2.2). Thus we have proved the main theorem.

3 Some remarks

Theorem 1 would be of no insterst unless there exists a definable real which makes $\aleph_1^L$ countable. The simplest way to make $\aleph_1^L$ countable is to add to L a generic function on ω onto $\aleph_1^L$ by forcing with finite partial functions. This forcing adds no ordinal-definable reals. Hence in the generic extension the non-constructible reals form a Π^1_2 sets of positive measure which does not contain any ordinal-definable real.

Much finer method to force $\aleph_1^L$ countable have been invented by Jensen and Solovay. In [3] they give a forcing notion $\mathcal{P} \in L$ and a Π^1_2 formula φ such that if $G \subset \mathcal{P}$ is generic then there exists a real $a \in V[G]$ such that

1. $L[a] \models (\forall x \subset \omega)[\, \varphi(x) \iff x = a\,]$;
2. every constructible real is recursive in a.

Clause 2 implies that the real a is non-constructible. Hence, in $L[a]$, a is a non-constructible Π^1_2 singleton. (See Theorem B of [1] for a yet sharper result along this line.)

Now let a be as above and $s = \mathcal{O}^a$, the hyperjump of a. That is to say, s is the set of notations of constructive ordinals relative to a. (See Chapter I of [4]. If you are not familiar with theory of hyperarithmetic hierarchy, you can use here the set $\{\, e \in \omega : \{e\}^a \in \mathbf{WO} \,\}$ instead of $\mathcal{O}^a$.) Since every ordinal below $\aleph^L_1$ is recursive-in-a, we have $\aleph^L_1 \leq \omega^a_1 < \omega^s_1$. In $L[a]$, on the other hand, s is a Π^1_2 singleton for, in $L[a]$,

$$x = s \iff (\forall y)[\, y = \{e_0\}^x \implies \varphi(y) \,\&\, x = \mathcal{O}^y \,],$$

where e_0 is a universal Gödel number which retrieves y from $\mathcal{O}^y$. Thus in the Jensen-Solovay model, there is a Π^1_2 singleton s such that $\aleph^L_1$ is a recursive-in-s ordinal:

Theorem 2 *There is a model of **ZFC** in which $0^\sharp$ does not exist while every Π^1_2 set of reals is Lebesgue measurable and every positive-measure Π^1_2 set contains Δ^1_3 members.*

In this model, however, exists a Δ^1_3 real r such that there exists a non-measurable $\Pi^1_2(r)$ set. Can we somehow multiply the Solovay-Jensen method to obtain an L-generic model of: *For every real r every $\Pi^1_2(r)$ set is Lebesgue measurable and if it has positive measure then it contains $\Delta^1_3(r)$ members?*

Our hypothesis of Theorem 1 "$\aleph^L_1$ is a recursive-in-s ordinal" seems quite essential, for otherwise $\mathbf{WO}(\aleph^L_1)$ is not a $\Sigma^1_1(s)$ set. We do not know whether this hypothesis can be weakened to "every ordinal below $\aleph^L_1$ is recursive in s," or equivalently, "every constructible real is $\Delta^1_1(s)$." Let us note here that this condition is strictly weaker than the one in Theorem 1:

Theorem 3 *There is a real $s \in 2^\omega$ in which every constructible real is recursive whereas $\aleph^L_1$ is not a recursive-in-s ordinal.*

Proof. A model $\mathcal{M} = (M, \in_M)$ of set theory is called an ω-model if all $\mathcal{M}$-integers are standard. Let us say an ω-model $\mathcal{M}$ to be nice if $M = \omega$ and the natural sequence $\langle\, (n)^{\mathcal{M}} : n \in \omega \,\rangle$ of the $\mathcal{M}$-integers is recursive in the real world. Every countable ω-model has an isomorphic copy which is nice.

Let $a \subset \omega$ be a real such that $\aleph^L_1 = \omega^a_1$. Then let Ψ be the set of reals $r \in 2^\omega$ which codes the $\in$-relation of a non-wellfounded nice ω-model of **KP** set theory in which an instance of a exists. Then Ψ is a non-empty $\Sigma^1_1(a)$ set. Therefore by the Gandy Basis Theorem (see, [4] Chap.III, 1.5), there is an $s \in \Psi$ such that $\omega^{\langle a,s\rangle}_1 = \omega^a_1 = \aleph^L_1$.

Let M be the model coded by s. Since M contais an instance of a, it follows that $\omega^a_1 \leq \omega^s_1$. Hence $\omega^s_1 = \aleph^L_1$. Each non-standard ordinal in M has order type $\omega^s_1 \times (1 + \mathrm{OrderType}(\mathbb{Q},<)) + \rho$ for some $\rho < \omega^s_1$. Therefore for each ordinal $\xi < \omega^s_1$ the set L_ξ is isomorphic to an initial part of the constructible hierarchy in M. It follows that M contains instances of all sets in $L_{\aleph_1^L}$. From this it follows that every constructible real is recursive in s. $\square$

References

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