Coanalytic sets with Borel sections.

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Fact. (Fujita and Mátrai) *Let $B \subset \boldsymbol{R} \times \boldsymbol{R}$ be a Borel set, such that horizontal section B^y is $\boldsymbol{\Sigma}^0_\alpha$ for every $y \in \boldsymbol{R}$. Then there is dense* G_δ *set $D \subset \boldsymbol{R}$ such that $B \cap (\boldsymbol{R} \times D)$ is $\boldsymbol{\Sigma}^0_\alpha \restriction (\boldsymbol{R} \times D)$.*

This can be proved by an straightforward induction using A. Louveau's solution of the *section problem* of Borel sets([**Lo**]). This Fact has been used in order to solve an old question by M. Laczkovich about differences of Borel measurable functions. (See [**FM**].)

Theorem. *The following statements are equivalent:*

(1) If $A \subset \boldsymbol{R} \times \boldsymbol{R}$ is $\boldsymbol{\Pi}^1_1$ and all the horizontal sections A^y are Borel, then there is a dense G_δ *set $D \subset \boldsymbol{R}$ such that $A \cap (\boldsymbol{R} \times D)$ is Borel;*

(2) similar, but A^y are $\boldsymbol{\Pi}^0_\alpha$ and $A \cap (\boldsymbol{R} \times D)$ is $\boldsymbol{\Pi}^0_\alpha \restriction (\boldsymbol{R} \times D)$, $(1 \leq \alpha < \omega_1)$;

(3) similar, but A^y are closed and $A \cap (\boldsymbol{R} \times D)$ is Borel;

(4) $\mathrm{BP}(\boldsymbol{\Sigma}^1_2)$*, [i.e., every $\boldsymbol{\Sigma}^1_2$ set of reals has the property of Baire.]* ◀

Proof. *From (1) to (2):* use the Fact.

From (2) to (3): immediate from the case $\alpha = 1$ of (2).

From (3) to (4): given $\boldsymbol{\Sigma}^1_2$ set $P \subset \boldsymbol{R}$, let $A \subset \boldsymbol{R} \times \boldsymbol{R}$ be $\boldsymbol{\Pi}^1_1$ such that $y \in P \iff \exists x[\langle x, y\rangle \in A]$. Uniformize A by a function $f : P \to \boldsymbol{R}$ with $\boldsymbol{\Pi}^1_1$ graph. Apply (3) to the graph of f. Then $P \cap D$ is $\boldsymbol{\Sigma}^1_1$ and D is co-meager. So P has BP.

From (4) to (1): this is the main part of today's talk...

Let $\mathbb{C}$ be the Cohen poset. Let $\mathrm{Cohen}(M)$ be the set of all $\mathbb{C}$-generic reals over the model M.

Lemma A. $\mathrm{BP}(\boldsymbol{\Sigma}^1_2)$ *if and only if* $\mathrm{Cohen}(L[r])$ *is co-meager for every $r \in \boldsymbol{R}$.* ◀

Let WO be the set of $w \in {}^\omega 2$ which codes a wellordering on ω. For each $w \in \mathrm{WO}$ let $\|w\|$ be the order-type (i.e., countable ordinal) that w codes.

Definition. *$X \subset \boldsymbol{R} \times \omega_1$ is Π^1_2* ***in the codes*** *if the set*

$$\left\{ \langle x, w\rangle \in \boldsymbol{R} \times {}^\omega 2 \;\middle|\; w \in \mathrm{WO},\ \langle x, \|w\|\rangle \in X \right\}$$

is (lightface) Π^1_2. ◀

Lemma B. *Let $X \subset \boldsymbol{R} \times \omega_1$ be Π^1_2 in the codes. Suppose that for every $y \in \boldsymbol{R}$ there is $\xi < \omega_1$ such that $\langle y, \xi\rangle \in X$. Then there is a countable δ such that for every $c \in \mathrm{Cohen}(L)$ there is $\xi < \delta$ such that $\langle c, \xi\rangle \in X$.* ◀

Proof of (4)⇒(1) [taking Lemmas for granted]. We put $\boldsymbol{R} = {}^{\omega}\omega$ and assume A is lightface Π^1_1. Let $f : \boldsymbol{R}\times\boldsymbol{R} \to {}^{\omega}2$ be a recursive function s.t. $A = f^{-1}[\mathrm{WO}]$.

Since A^y is Borel, the image $f[A^y \times \{y\}]$ is bounded in WO, that is to say,

$$\forall y \in \boldsymbol{R} \exists \xi < \omega_1 \forall x \Big[\langle x, y\rangle \in A \implies \|f(x,y)\| < \xi \Big].$$

For each $\xi < \omega_1$ set

$$\mathrm{WO}_\xi = \Big\{ w \in \mathrm{WO} \;\Big|\; \|w\| < \xi \Big\}$$

and let

$$X = \Big\{ \langle y, \xi\rangle \;\Big|\; f[A^y \times \{y\}] \subset \mathrm{WO}_\xi. \Big\}$$

Obsetve that X is Π^1_2 in the codes. Applying Lemma B we find $\delta < \omega_1$ such that

$$\forall c \in \mathrm{Cohen}(L) \exists \xi < \delta \Big[\langle c, \xi\rangle \in X \Big].$$

Then we have

$$A \cap \big(\boldsymbol{R} \times \mathrm{Cohen}(L)\big) = f^{-1}[\mathrm{WO}_\delta] \cap \big(\boldsymbol{R} \times \mathrm{Cohen}(L)\big).$$

By Lemma A there is a dense G_δ set $D \subset \mathrm{Cohen}(L)$. ◀

Proof of Lemma B. Let $\varphi(y, w)$ be a Π^1_1 formula such that

$$\begin{aligned} \langle y, \xi\rangle \in X &\iff \exists w \in \mathrm{WO}\Big(\xi = \|w\| \wedge \varphi(y,w) \Big) \\ &\iff \forall w \in \mathrm{WO}\Big(\xi = \|w\| \implies \varphi(y,w). \Big) \end{aligned}$$

Then we have, by assumption of the lemma,

$$(*) \qquad \forall y \exists \xi < \omega_1 \forall w \Big(w \in \mathrm{WO} \wedge \|w\| = \xi \implies \varphi(y,w) \Big)$$

let $\varphi^*(y,\xi)$ stand for "$\forall w \cdots$" part of (*). Then $\varphi^*(y,\xi)$ is absolute for every proper class model in which ξ is countable.

Let $c \in \mathrm{Cohen}(L)$. Suppose that $\langle c, \xi\rangle \in X$.

Let $g : \omega \to \xi$ be $\mathrm{Coll}(\xi)$-generic over $L[c]$. Then

$$L[c,g] \models \varphi^*(c,\xi)$$

so that there are forcing conditions $p \in \mathbb{C}$ and $q \in \mathrm{Coll}(\xi)$ such that c meets p, g meets q and

$$\big\langle p, q\big\rangle \Vdash_{(\mathbb{C}\times\mathrm{Coll}(\xi))} L[\dot{c}, \dot{g}] \models \varphi^*(\dot{c}, \check{\xi}).$$

Then by absoluteness of forcing relations,

$$L \models \Big(\big\langle p, q\big\rangle \Vdash_{(\mathbb{C}\times\mathrm{Coll}(\xi))} \varphi^*(\dot{c}, \check{\xi}) \Big).$$

By homogeneity of the poset Coll(ξ),

$$L \models \Big(\langle p, \emptyset \rangle \Vdash_{(\mathbb{C} \times \mathrm{Coll}(\xi))} \varphi^*(\dot{c}, \check{\xi}) \Big).$$

where $\emptyset$ is the largest member of Coll(ξ).

For each $\xi < \omega_1$ let

$$Y_\xi = \Big\{ p \in \mathbb{C} \;\Big|\; L \models \big(\langle p, \emptyset \rangle \Vdash_{(\mathbb{C} \times \mathrm{Coll}(\xi))} \varphi^*(\dot{c}, \check{\xi}) \big) \Big\}.$$

Then $\bigcup_{\xi < \omega_1} Y_\xi$ is pre-dense in $\mathbb{C}$. By ccc, there is $\delta < \omega_1$ such that $\bigcup_{\xi < \delta} Y_\xi$ is already pre-dense in $\mathbb{C}$. ◀

Daiske Ikegami observed that $\mathbb{C}$ in Lemma B can be replaced by other ccc forcing notions that is (lightface) Σ^1_1 and strongly arboreal. Daisuke also pointed out that Sacks forcing does not satisfy Lemma B nor clause (3) of Theorem.

By Montgomery's result on the category quantifier, we obtain

Corollary. *Assume* BP($\boldsymbol{\Sigma}^1_2$). *Let* $A \subset \boldsymbol{R} \times \boldsymbol{R}$ *be* $\boldsymbol{\Pi}^1_1$ *such that* A^y *is* $\boldsymbol{\Sigma}^0_\alpha$ *for every* $y \in \boldsymbol{R}$. *Then*

$$\exists^* A = \{ x \in \boldsymbol{R} \mid A_x \text{ is not meager} \}$$

is $\boldsymbol{\Sigma}^0_\alpha$. ◀

Question. *Does this statement imply* BP($\boldsymbol{\Sigma}^1_2$)*?*

References

[**FM**] H. Fujita and T. Mátrai, *On the difference property of Borel measurable functions*, submitted (August 2008). Available at authors' websites.

[**Lo**] A. Louveau, *A separation theorem for* Σ^1_1, Trans. Amer. Math. Soc. **260** (1980), 363–378.