On the difference property of Borel measurable functions

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Abstract

If an atomlessly measurable cardinal exists, then the class of Lebesgue measurable functions, the class of Borel functions, and the Baire classes of all order have the difference property. This gives a consistent positive answer to Laczkovich's Problem 2 posed in [12]. We also give a complete positive answer to Laczkovich's Problem 3 in [12] concerning Borel functions with Baire-α differences.

1 Introduction

For each real function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ and each real constant $h \in \boldsymbol{R}$, the difference function $x \mapsto f(x+h) - f(x)$ is denoted by $\Delta_h f$. If a class $\mathcal{F} \subseteq {}^{\boldsymbol{R}}\boldsymbol{R}$ forms a translation invariant vector space over $\boldsymbol{R}$, then every $f \in \mathcal{F}$ satisfies the condition $\forall h \in \boldsymbol{R}\ \left[\Delta_h f \in \mathcal{F}\right]$.

If moreover $\mathcal{F}$ contains nonzero constant functions then every function of the form $f = g + \theta$, where g is in $\mathcal{F}$ and θ is *additive* (i.e., $\theta(x+y) = \theta(x) + \theta(y)$ holds for every $x, y \in \boldsymbol{R}$), satisfies the same condition. The difference property is the converse of this trivial observation.

Definition 1.1. *A class $\mathcal{F} \subseteq {}^{\boldsymbol{R}}\boldsymbol{R}$ of real functions is said to have the* difference property *if every function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ satisfying $\forall h \in \boldsymbol{R}\ \left[\Delta_h f \in \mathcal{F}\right]$ has the form $f = g + \theta$ where $g \in \mathcal{F}$ and θ is additive.*

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The notion was invented by N.G. de Bruijn who proved that the class $C(\boldsymbol{R})$ of all continuous real functions has the difference property (see [2]). In fact, various subclasses of $C(\boldsymbol{R})$, which play important role in classical analysis, have the difference property: for example, the class of differentiable functions, the class $C^r(\boldsymbol{R})$ of functions with continuous rth derivatives, the class of real analytic functions, etc. These results are reviewed in M. Laczkovich's survey paper [14].

On the other hand, it was pointed out by P. Erdős that the second Baire class does not have the difference property if we assume the Continuum Hypothesis (CH). Under CH there exists a set $A \subseteq \boldsymbol{R}$ such that for every $h \in \boldsymbol{R}$ the difference $(A+h) \setminus A$ is countable and yet A is *not* Lebesgue measurable. The characteristic function of such a set A has Baire-2 difference functions but it is not the sum of a Lebesgue measurable function and an additive function. Thus the difference property of the class of Lebesgue measurable functions or the class of Borel functions cannot be established by the conventional ZFC axioms of set theory.

The consistency of the difference property of the class $\mathcal{L}$ of Lebesgue measurable functions has been established by Laczkovich in [12] and [13]. In [12] Laczkovich proves that $\mathcal{L}$ has the *weak difference property* in the sense that every function f satisfying $\forall h \in \boldsymbol{R}\ \left[\Delta_h f \in \mathcal{L}\right]$ is of the form $f = g + \theta + S$ where $g \in \mathcal{L}$, θ is additive and S is *small*, i.e., it satisfies $\Delta_h S(x) = 0$ a.e. for every $h \in \boldsymbol{R}$. By this result, the consistency of the difference property of $\mathcal{L}$ is reduced to the problem when is every small function Lebesgue measurable, which turns out to be a kind of strong Fubini theorem in which the measurability condition on the function of two variables is considerably relaxed. In [13], such a strong Fubini theorem is shown to be consistent with ZFC (see Section 5.1 for the precise statement). Walking along the same line, in Section 3 we will prove the following theorem, which establishes the difference property of $\mathcal{L}$ assuming the existence of an atomlessly measurable cardinal.

Theorem 1.2. *Assume an atomlessly measurable cardinal exists. Then the class of Lebesgue measurable functions has the difference property.*

The above mentioned work of Laczkovich on Lebesgue measurable functions leaves open the consistency of the difference property of Borel functions. In [12], Laczkovich posed the following three problems motivated by the example of Erdős.

Problem 1. *Does the first Baire class $\mathcal{B}_1$ have the difference property?*

Problem 2. *Suppose that $\Delta_h f$ is Borel for every $h \in \boldsymbol{R}$. Is there a countable ordinal α such that $\Delta_h f$ is Baire-α for all $h \in \boldsymbol{R}$?*

Problem 3. *Suppose that f is Borel and for $\Delta_h f$ is Baire-α for every $h \in \boldsymbol{R}$. Then is f itself Baire-α?*

Problem 1 has been solved positively by Laczkovich himself. See [14, Section 7]. One finds there that some useful subclasses of $\mathcal{B}_1$ have the difference property: approximately continuous functions, derivatives, Darboux Baire-1 functions, etc.

A counter-example to Problem 2 has been given by R. Filipów and I. Recław assuming CH (see [5, Theorem 3.1]). Their use of CH is unavoidable, though it can be replaced by a version of the Covering Property Axioms (see [4, Corollary 5.1.11]). In Section 3 we will prove that the existence of an atomlessly measurable cardinal excludes such counter-examples; moreover, an atomlessly measurable cardinal implies that the class of Borel functions has the difference property.

Theorem 1.3. *Assume an atomlessly measurable cardinal exists. Then the class of Borel functions has the difference property. Moreover, for every $\alpha < \omega_1$, the class of Baire-α functions has the difference property.*

In Section 4, we will prove the following result, which provides a complete positive answer to Problem 3.

Theorem 1.4. *Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a Borel function and let α be a countable ordinal. Suppose that for every $h \in \boldsymbol{R}$ the difference function $\Delta_h f$ is of Baire class α. Then f itself is of Baire class α.*

Partial answers to Problem 3 were known earlier: Laczkovich in [14, Section 7 p. 391] proved the statement for every *bounded* Borel function, while in [7] the first author solved the problem for every $\alpha \geq \omega$. Actually, it is the solution of Laczkovich for bounded functions which give us the impression that the difference property of the Borel functions should be consistent. However, our solution of Problem 3 goes in a quite different way from the approach of Laczkovich in [14] or the first author's in [7]. We will use a Baire category argument while previous results used measure theoretic methods. These three attempts are only loosely related to each other by the basic observation that questions about the difference property usually reduce to appropriate extensions of section results like e.g., the Baire-Namioka theorem, the Fubini theorem, the Kuratowski-Ulam theorem, etc.

We collect in Section 2 the preliminary results that we need. After obtaining our main results, in Section 5 we will discuss their possible generalizations under appropriate set-theoretic assumptions.

2 Preliminaries

The power set of X is denoted by $\mathcal{P}(X)$. The cardinality of the continuum is denoted by $\mathfrak{c}$.

2.1 Sections of sets and functions

Let X and Y be sets and let $A \subseteq X \times Y$ be a set of pairs. The *vertical section* of A at $x \in X$ is the set $A_x = \{y \in Y : \langle x, y\rangle \in A\}$. Similarly, the *horizontal section* of A at $y \in Y$ is the set $A^y = \{x \in X : \langle x, y\rangle \in A\}$. For a function $F : X \times Y \to Z$ of two variables, we also define the vertical sections $F_x : Y \to Z$ $(x \in X)$ and the horizontal sections $F^y : X \to Z$ $(y \in Y)$ by $F_x(y) = F^y(x) = F(x, y)$ $(x \in X, y \in Y)$.

2.2 Basic notions from descriptive set theory

Our reference for basic notions from descriptive set theory is [11]. The αth additive (resp. multiplicative) class of the Borel hierarchy is denoted by $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$). In particular, $\mathbf{\Sigma}^0_1$ denotes the class of open sets, $\mathbf{\Pi}^0_1$ the closed sets, $\mathbf{\Sigma}^0_2$ the F_σ sets, etc. We define the αth ambiguous class by $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. The class of all Borel sets is denoted by $\boldsymbol{B}$.

The nth additive (resp. multiplicative) class of the projective hierarchy is denoted by $\mathbf{\Sigma}^1_n$ (resp. $\mathbf{\Pi}^1_n$). Therefore $\mathbf{\Sigma}^1_1$ denotes the class of analytic sets, $\mathbf{\Pi}^1_1$ the coanalytic sets, etc. We define the nth ambiguous class by $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

When we say a set A is in one of the above defined pointclasses, we assume we know the space which A is a subset of. If we need to specify the space we are dealing with, we write $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^1_n(Y)$, etc.

Let X and Y be Polish spaces and let $\mathbf{\Gamma}$ be a pointclass. We say a function $f : X \to Y$ is $\mathbf{\Gamma}$-*measurable* if for every open set $O \subseteq Y$ we have $f^{-1}[O] \in \mathbf{\Gamma}$. Therefore a function is $\mathbf{\Sigma}^0_1$-measurable if and only if it is continuous. A classical result, due to Lebesgue, says that a function from a Polish space X to a Polish space Y is of Baire class α if and only if it is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable (see e.g. [11, (24.3) Theorem p. 190]).

One can endow every $\mathbf{\Pi}^0_2$ subset of a Polish space with a complete metric without changing topology; conversely, every completely metrizable subset of a metric space is G_δ. For the details, we refer to [11, (3.11) Theorem p. 17]

2.3 Universal measures.

A measure space of the form $(Y, \mathcal{P}(Y), \mu)$ (i.e. where μ is defined for *every* subset of Y) is called a *universal* measure space.

It is fairly easy to construct on any set a universal probability measure space which is concentrated on a countable set of point masses (a *point mass* is a point with nonzero measure.) On the other hand, the existence of a universal probability measure space without point masses is not provable in ZFC. The existence of such universal measure is equivalent to the existence of a *real-valued measurable cardinal*, which is either two-valued measurable or atomlessly measurable.

Two-valued measurable cardinals are those cardinals that are usually called *measurable* cardinals in conventional terminology of set theory. They play a central role in the study of large cardinals.

An *atomlessly measurable cardinal* is an uncountable cardinal κ carrying a universal probability measure space $(\kappa, \mathcal{P}(\kappa), \mu)$ which is κ*-additive*, i.e. the union of fewer than κ μ-null sets is μ-null, and *atomless*, i.e. the whenever $\mu(A) > 0$ then there is $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$. It is known that any atomlessly measurable cardinal is less than or equal to $\mathfrak{c}$. The abbreviation RVMC stands for the statement *there exists an atomlessly measurable cardinal.*

It is known that RVMC is consistent with ZFC if and only if so is the existence of a two-valued measurable cardinal. For more information about atomlessly measurable cardinals, including the equiconsistency result, we refer the reader to [6] and [19].

It is clear from the definition that the smallest cardinal κ such that μ is not κ-additive is a successor cardinal. Thus we can define $\mathrm{add}(\mu)$ to be the largest cardinal κ such that μ is κ-additive. In other words, $\mathrm{add}(\mu)$ is the smallest possible size of a family $\mathcal{A}$ of μ-null sets such that $\bigcup \mathcal{A}$ is not μ-null.

Lemma 2.1. *Every σ-finite universal measure is ω_2-additive.*

Proof. Since μ is σ-finite, there is a partition $(Y_n)_{n<\omega}$ of Y such that for every $n < \omega$, $\mu(Y_n) < \infty$. It is enough to verify that μ is ω_2-additive on each Y_n separately, so we can assume $Y = Y_n$ for some $n < \omega$. We consider only the non-trivial $\mu(Y) > 0$ case.

Set $\kappa = \mathrm{add}(\mu)$. By definition, there is a family $(Y_\alpha)_{\alpha<\kappa}$ of pairwise disjoint subsets of Y such that for every $\alpha < \kappa$, $\mu(Y_\alpha) = 0$ but $\mu(\bigcup_{\alpha<\kappa} Y_\alpha) > 0$. Let $I = \{A \subseteq \kappa\colon \mu(\bigcup_{\alpha\in A} Y_\alpha) = 0\}$. Then I is a κ-complete σ-saturated ideal on κ. So by [9, Lemma 10.14 p. 132], κ cannot be a successor cardinal, in particular $\kappa > \omega_1$. $\square$

2.4 Product σ-algebras

For a pair of σ-algebras $\mathcal{A}$ and $\mathcal{B}$ on sets X and Y respectively, let $\mathcal{A}\otimes\mathcal{B}$ be the σ-algebra on $X \times Y$ generated by $\{A \times B \colon A \in \mathcal{A},\ B \in \mathcal{B}\}$. This is the smallest σ-algebra on $X \times Y$ that makes the projections $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ measurable. We call $\mathcal{A} \otimes \mathcal{B}$ *the product σ-algebra* of $\mathcal{A}$ and $\mathcal{B}$.

It is easy to see that for every $A \in \mathcal{A}\otimes\mathcal{B}$ and every $\langle x, y\rangle \in X \times Y$, the horizontal section A^y and the vertical section A_x are in $\mathcal{A}$ and $\mathcal{B}$ respectively. The converse is not true in general:

- The natural ordering relation $<$ on ω_1, as a set of pairs, has the property that every horizontal section is countable and every vertical section is co-countable. So if $\mathcal{A}$ is the countable/co-countable algebra on ω_1, all horizontal and vertical sections of $<$ belong to $\mathcal{A}$. However, no ordering relation belongs to the product $\mathcal{A} \otimes \mathcal{A}$ because relations in $\mathcal{A} \otimes \mathcal{A}$ can never be anti-symmetric.

- If $\kappa > \mathfrak{c}$ then $\mathcal{P}(\kappa \times \kappa) \neq \mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa)$ because the diagonal set does not belong to the right hand side.

However, as an important special case, the following is true.

Lemma 2.2. *Let X be a separable metric space and Y be any nonempty set. Let $A \subseteq X \times Y$. Suppose that there is a countable ordinal α such that for every $y \in Y$ the horizontal section A^y is in $\mathbf{\Sigma}^0_\alpha(X)$. Then $A \in \boldsymbol{B}(X)\otimes\mathcal{P}(Y)$.*

Proof. Let first $\alpha = 1$. Let $\{N_i \colon i < \omega\}$ be an open basis in X. For every $i < \omega$, define $B_i = \{y \in Y \colon N_i \subseteq A^y\}$. Then it is routine to check that $A = \bigcup_{i\in\omega}(N_i \times B_i)$.

The $\alpha > 1$ case is a straightforward induction on α. □

Note that the converse of Lemma 2.2 is also true. Therefore the assumption $A^y \in \mathbf{\Sigma}^0_\alpha$ cannot be weakened to $A^y \in \boldsymbol{B}(X)$ unless Y is countable.

2.5 The Fubini Theorem

For a pair of σ-finite measure spaces, say $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \mu)$, the product measure on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ will be denoted by $\lambda \otimes \mu$. We will need the following versions of the Fubini Theorem (see e.g. [18, 8.8 Theorem p. 164] or [17, Theorem 1 p. 325, Theorem 2 p. 329]).

Proposition 2.3. (The Fubini Theorem) *Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \mu)$ be σ-finite measure spaces. If a function $u \colon X \times Y \to \boldsymbol{R}$ is $\lambda \otimes \mu$-integrable then* *(i) the vertical section function* $u_x\colon y \mapsto u(x,y)$ *is* μ*-integrable for* λ*-almost every* x*,*

(ii) the function $x \mapsto \int_Y u_x(y)d\mu(y)$ *is* $\mathcal{A}$*-measurable,*

(iii) $\iint_{X\times Y} u(x,y)d(\lambda\otimes\mu)(x,y) = \int_X \left(\int_Y u_x(y)d\mu(y)\right) d\lambda(x)$ *holds;*

and similarly for the horizontal sections.

Proposition 2.4. (The Fubini-Tonelli Theorem) *Let* $(X,\mathcal{A},\lambda)$ *and* $(Y,\mathcal{B},\mu)$ *be* σ*-finite measure spaces. Let* $u\colon X\times Y \to \boldsymbol{R}$ *be an* $\mathcal{A}\otimes\mathcal{B}$*-measurable function such that at least one of the iterated integrals*

$$\int_X \left(\int_Y |u_x(y)|d\mu(y)\right) d\lambda(x) \quad \text{and} \quad \int_Y \left(\int_X |u^y(x)|d\lambda(x)\right) d\mu(y)$$

exists. Then u *is* $\lambda\otimes\mu$*-integrable and the consequences of Proposition 2.3 follow.*

For more on these results we refer the reader to textbooks on integrals, e.g., Halmos [8, Ch. VII], Taylor [20, Ch. 6] or Yeh [21, §23].

2.6 Category quantifiers

The σ-ideal of meager (i.e., of first category) sets in a topological space X is denoted by $\mathcal{M}(X)$. We often write just $\mathcal{M}$ when the space X is clear from the context.

Let X and Y be Polish spaces. Let $A \subseteq X\times Y$. We define two subsets $\exists^{*Y}A$ and $\forall^{*Y}A$ of X by

$$\exists^{*Y}A = \{x\in X\colon A_x \notin \mathcal{M}(Y),\}$$
$$\forall^{*Y}A = \{x\in X\colon Y\setminus A_x \in \mathcal{M}(Y)\}.$$

In Section 4 we will need the following result due to Montgomery (see e.g [11, (22.22) Exercise p. 174]).

Lemma 2.5. *Let* X *and* Y *be Polish spaces. Let* α *be a countable nonzero ordinal and let* $B\subseteq X\times Y$ *be a* $\mathbf{\Sigma}^0_\alpha(X\times Y)$ *set. Then* $\exists^{*Y}B$ *is a* $\mathbf{\Sigma}^0_\alpha(X)$ *set. Similarly, if* B *is* $\mathbf{\Pi}^0_\alpha(X\times Y)$*, then* $\forall^{*Y}B$ *is* $\mathbf{\Pi}^0_\alpha(X)$*.*

2.7 Borel sets with $\mathbf{\Sigma}^0_\alpha$ sections

Let X and Y be Polish spaces. For every countable nonzero ordinal α, let $\mathcal{S}^\Sigma_\alpha$ be the set of Borel sets $B\subseteq X\times Y$ such that for every $y\in Y$ the horizontal section B^y is in $\mathbf{\Sigma}^0_\alpha(X)$. Let $\mathcal{T}^\Pi_0 = \{U\times B\colon U\in\mathbf{\Sigma}^0_1(X), B\in\boldsymbol{B}(Y)\}$. For every countable nonzero ordinal α, we define $\mathcal{T}^{\Sigma}_{\alpha}$ and $\mathcal{T}^{\Pi}_{\alpha}$ by the following induction:

$$\mathcal{T}^{\Sigma}_{\alpha} = \bigvee^{\omega}\left(\bigcup_{\beta<\alpha}\mathcal{T}^{\Pi}_{\beta}\right), \qquad \mathcal{T}^{\Pi}_{\alpha} = \{B \subseteq X \times Y : (X \times Y) \setminus B \in \mathcal{T}^{\Sigma}_{\alpha}\}$$

where $\bigvee^{\omega}$ is the closure under countable unions. It is clear that $\boldsymbol{B}(X\times Y) = \bigcup_{\alpha<\omega_1}\mathcal{T}^{\Sigma}_{\alpha} = \bigcup_{\alpha<\omega_1}\mathcal{T}^{\Pi}_{\alpha}$. It is also clear that $\mathcal{T}^{\Sigma}_{\alpha} \subseteq \mathcal{S}^{\Sigma}_{\alpha}$. The following proposition, a subtle result due to A. Louveau, says the converse is also true (see [16, Theorem 1 p. 375]).

Proposition 2.6. *For every countable nonzero ordinal α, $\mathcal{T}^{\Sigma}_{\alpha} = \mathcal{S}^{\Sigma}_{\alpha}$.*

Proposition 2.6 will play an important role in Section 4. We refer to [16, §3] for more about it.

3 Difference property of Lebesgue measurable functions and Borel functions under RVMC

In this section we prove Theorem 1.2 and Theorem 1.3. These results should be contrasted with the earlier results mentioned in the introduction showing that under CH none of the classes between the Baire class 2 and the Lebesgue measurable functions has difference property.

3.1 Lemmas on measurability of integrals

In order to prove these theorems we need several lemmas asserting the preservation of the measurability of functions under integration. Our first lemma is due to Laczkovich and Miller [15, Lemma 6].

Lemma 3.1. *Let X be a Polish space and $(Y, \mathcal{P}(Y), \mu)$ be a universal probability measure space. Let α be a countable ordinal. Let $F\colon X \times Y \to \boldsymbol{R}$ be a bounded function such that for every $y \in Y$ the horizontal section $F^y\colon x \mapsto F(x, y)$ is of Baire class α. Then the function*

$$x \mapsto \int_Y F(x, y)d\mu(y)$$

is also of Baire class α.

Lemma 3.2. *Let X be a Polish space and $(Y, \mathcal{P}(Y), \mu)$ be a universal probability measure space. Let $F\colon X \times Y \to \boldsymbol{R}$ be a bounded function such that* *for every $y \in Y$ the horizontal section $F^y\colon x \mapsto F(x,y)$ is Borel. Then the function*

$$x \mapsto \int_Y F(x,y)d\mu(y)$$

is Borel.

Proof. For each countable ordinal α, let Y_α be the set of $y \in Y$ for which F^y is of Baire class α. Then we have $Y_\alpha \subseteq Y_\beta$ $(\alpha \leq \beta < \omega_1)$ and $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$. It follows that $\mu(Y_\alpha) \leq \mu(Y_\beta)$ $(\alpha \leq \beta < \omega_1)$. Since there is no strictly increasing ω_1-sequence of real numbers, there is an $\alpha < \omega_1$ such that $\mu(Y_\alpha) = \mu(Y_\beta)$ $(\alpha \leq \beta < \omega_1)$.

By Lemma 2.1, μ is ω_2-additive. So we have $\mu(Y_\alpha) = \mu(Y) = 1$. Therefore, for every $x \in X$ we have

$$\int_Y F(x,y)d\mu(y) = \int_{Y_\alpha} F(x,y)d\mu(y).$$

By Lemma 3.1 for the universal probability measure space $(Y_\alpha, \mathcal{P}(Y_\alpha), \mu)$, we get that the function $x \mapsto \int_Y F(x,y)d\mu(y)$ is of Baire class α. □

Lemma 3.3. *Let X be a Polish space and $(Y, \mathcal{P}(Y), \mu)$ be a σ-finite universal measure space. Let $F\colon X \times Y \to \boldsymbol{R}$ be such that for every $y \in Y$ the horizontal section $F^y\colon X \to \boldsymbol{R}$ is Borel and for every $x \in X$ the vertical section F_x is μ-integrable. Then the function*

$$x \mapsto \int_Y F(x,y)d\mu(y)$$

is Borel.

Proof. Since μ is σ-finite, we can find $Y_n \subseteq Y$ with $\mu(Y_n) < \infty$ $(n < \omega)$ such that $Y_n \subseteq Y_m$ for $n \leq m$ and $Y = \bigcup_{n<\omega} Y_n$.

For each positive integer N, let

$$F_N(x,y) = \begin{cases} N, & \text{if } F(x,y) \geq N, \\ F(x,y), & \text{if } -N < F(x,y) < N, \\ -N, & \text{if } F(x,y) \leq -N. \end{cases}$$

Then F_N is bounded and the horizontal section $(F_N)^y$ is Borel for every $y \in Y$. Therefore by Lemma 3.2 the function

$$f_N(x) = \int_{Y_N} F_N(x,y)d\mu(y)$$

is Borel. Since the vertical section F_x is μ-integrable for every $x \in X$ we have

$$\lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} \int_{Y_N} F_N(x,y)d\mu(y) = \int_Y F(x,y)d\mu(y).$$

So the function $x \mapsto \int_Y F(x,y)d\mu(y)$ is the pointwise limit of a sequence of Borel functions hence it is Borel, as required. $\square$

Lemma 3.4. *Let X be a Polish space and let λ be a σ-finite Borel measure on X. Let $(Y, \mathcal{P}(Y), \mu)$ be a σ-finite universal measure space. Let $F\colon X \times Y \to \boldsymbol{R}$ be such that for every $y \in Y$ the horizontal section $F^y\colon X \to \boldsymbol{R}$ is λ-measurable in the usual sense and for every $x \in X$ the vertical section F_x is μ-integrable. Then the function*

$$x \mapsto \int_Y F(x,y)d\mu(y)$$

is λ-measurable.

Proof. By Luzin's Theorem (see e.g. [11, (17.12) Theorem p. 108]), there is a function $G\colon X \times Y \to \boldsymbol{R}$ such that for every $y \in Y$ the horizontal section G^y is $\boldsymbol{\Sigma}^0_3$-measurable and $\{x\colon G^y(x) \neq F^y(x)\}$ is λ-null. Then, by Lemma 2.2, G is $\boldsymbol{B}(X) \otimes \mathcal{P}(Y)$-measurable. Let $E = \{\langle x,y\rangle\colon G(x,y) \neq F(x,y)\}$. Let $D \subseteq X \times Y$ be such that D^y is a λ-null $\boldsymbol{\Pi}^0_2$ set containing E^y $(y \in Y)$. Then $D \in \boldsymbol{B}(X) \otimes \mathcal{P}(Y)$ by Lemma 2.2. So by the Fubini Theorem we have

$$\begin{aligned}(\lambda \otimes \mu)(D) &= \iint_{X\times Y} \chi_D(x,y)d(\lambda \otimes \mu)(x,y)\\ &= \int_Y \left(\int_X \chi_D(x,y)d\lambda(x)\right) d\mu(y)\\ &= \int_Y \lambda(D^y)d\mu(y)\\ &= 0.\end{aligned}$$

From this it follows that $\mu(\{y\colon F_x(y) \neq G_x(y)\}) = 0$ for λ-almost every x. So for λ-almost every x, the section G_x is μ-integrable and

$$\int_Y F(x,y)d\mu(y) = \int_Y G(x,y)d\mu(y).$$

By Lemma 3.3 the function $x \mapsto \int_Y G(x,y)d\mu(y)$ is Borel. Therefore the function $x \mapsto \int_Y F(x,y)d\mu(y)$ is λ-measurable, as required. $\square$

3.2 Proof of Theorem 1.2.

Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a function such that the difference function $\Delta_h f$ is Lebesgue measurable for every $h \in \boldsymbol{R}$. By a result of Laczkovich [12, Theorem 3], $f = g + \theta + S$ with a Lebesgue measurable g, an additive θ, and a function S such that for every $h \in \boldsymbol{R}$ the set $\{x\colon \Delta_h S(x) \neq 0\}$ is Lebesgue null. Therefore it is sufficient to show that every such a function S is Lebesgue measurable.

Let κ be an atomlessly measurable cardinal. Then $\kappa \leq \mathfrak{c}$ and the Lebesgue measure λ can be extended to a κ-additive universal measure (see [6, Theorem 1D(e)], [19], or [10, Section 2] for the details). So let μ be a κ-additive universal measure on $\boldsymbol{R}$ that extends the Lebesgue measure.

Let $F(x,y) = S(x+y) - S(x) - S(y)$. Then for every y we have $F^y(x) = -S(y)$ for λ-almost every x. Similarly, for every x we have $F_x(y) = -S(x)$ for μ-almost every y. In particular, $F_x|_{[0,1]}$ is μ-integrable for every $x \in \boldsymbol{R}$. By Lemma 3.4 we have

$$x \mapsto \int_{[0,1]} F(x,y) d\mu(y) = -S(x)$$

is λ-measurable, i.e., Lebesgue measurable. Therefore S is Lebesgue measurable, as required.

3.3 Proof of Theorem 1.3.

Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a function such that the difference functions $\Delta_h f$ are Borel for every $h \in \boldsymbol{R}$. By Theorem 1.2, $f = g + \theta$ with Lebesgue measurable g and additive θ. Let $\varphi\colon \boldsymbol{R} \to \boldsymbol{R}$ be a Borel function such that $g(x) = \varphi(x)$ for almost every x. Let $S = g - \varphi$. Then for every $h \in \boldsymbol{R}$, $\Delta_h S$ is a Borel function and $\Delta_h S(x) = 0$ for almost every x.

In order to show that every such a function S is Borel, we proceed as in the proof of Theorem 1.2 but this time we apply Lemma 3.3 rather than Lemma 3.4.

The difference property for the Baire class α functions follows from the difference property of the Borel functions through Theorem 1.4.

4 Borel functions with Baire α differences

This section is devoted to the proof of Theorem 1.4, which gives an affirmative answer to [12, Problem 3] of M. Laczkovich.

As we have mentioned in the Introduction, unlike the results of Section 3 which use measure theory conceptually as well as technically, the proof we present in this section for Theorem 1.4 uses only Baire category. However, at the end of this section we sketch an alternative proof for Theorem 1.4 which is of measure theoretic nature. We will discuss the interplay of different proofs with possible consistent generalizations of our results in Section 5.

We will need the following folklore lemma. For every $x \in \boldsymbol{R}$ and $\varepsilon > 0$, let $B(x,\varepsilon) = \{y \in \boldsymbol{R}\colon |y - x| < \varepsilon\}$.

Lemma 4.1. *Let X be a Polish space and let $f\colon X \to \boldsymbol{R}$ be a Baire measurable function. Then there exists a comeager $\boldsymbol{\Pi}^0_2(X)$ set G and a function $g\colon X \to \boldsymbol{R}$ of Baire class 1 such that $f|_G = g|_G$.*

Proof. By Nikodým's Theorem (see e.g. [11, (8.38) Theorem p. 52]), there exists a comeager $\boldsymbol{\Pi}^0_2(X)$ set G such that $f|_G$ is continuous. We define $h\colon X \to \boldsymbol{R} \cup \{\pm\infty\}$ by

$$h(x) = \lim_{\varepsilon \to +0} \big(\inf\{f(y) \colon y \in B(x,\varepsilon) \cap G\}\big) \qquad (x \in X).$$

By the continuity of $f|_G$, $h|_G = f|_G$. For every $c \in \boldsymbol{R}$ the set $\{x \in X \colon h(x) > c\}$ is open. From this it follows that h is a function of Baire class 1 and the set $h^{-1}(\{\pm\infty\})$ is $\boldsymbol{\Pi}^0_2(X)$. Since $h^{-1}(\{\pm\infty\})$ and G are disjoint $\boldsymbol{\Pi}^0_2$ sets, by the separation principle there exists a $\boldsymbol{\Delta}^0_2(X)$ set D such that $h^{-1}(\{\pm\infty\}) \subseteq D$ and $G \cap D = \emptyset$. We define $g\colon X \to \boldsymbol{R}$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in X \setminus D; \\ 0 & \text{if } x \in D. \end{cases}$$

Then g is a function of Baire class 1 and $g|_G = f|_G$, as required. □

Our main lemma is the following. For every topological space Y and $A \subseteq Y$, $\mathrm{cl}_Y(A)$ denotes the closure of A.

Lemma 4.2. *Let X and Y be Polish spaces. Let α be a countable nonzero ordinal. Let $B \subseteq X \times Y$ be a Borel set such that the horizontal section B^y is in $\boldsymbol{\Sigma}^0_\alpha(X)$ for every $y \in Y$. Then there is a comeager $\boldsymbol{\Pi}^0_2(Y)$ set G such that $B \cap (X \times G)$ is in $\boldsymbol{\Sigma}^0_\alpha(X \times G)$.*

Proof. First we show that for every set $A \subseteq Y$ with the Baire property, there is a comeager $\boldsymbol{\Pi}^0_2(Y)$ set $\mathcal{G}(A)$ such that $A \cap \mathcal{G}(A)$ is relatively clopen in $\mathcal{G}(A)$. By [11, (8.23) Proposition p. 47], for every A with the Baire property there is an open set $\mathcal{O}(A) \subseteq Y$ such that $(A \setminus \mathcal{O}(A)) \cup (\mathcal{O}(A) \setminus A)$ is meager. Let $\mathcal{E}(A)$ be a meager $\boldsymbol{\Sigma}^0_2(Y)$ set containing the meager sets $(A \setminus \mathcal{O}(A))$, $(\mathcal{O}(A) \setminus A)$ and $\mathrm{cl}_Y(\mathcal{O}(A)) \setminus \mathcal{O}(A)$. Let $\mathcal{G}(A) = Y \setminus \mathcal{E}(A)$; we show that this definition fulfills the requirements.

It is immediate that $\mathcal{G}(A)$ is a comeager $\boldsymbol{\Pi}^0_2(Y)$ set. By $A = (A \cap \mathcal{O}(A)) \cup (A \setminus \mathcal{O}(A))$ we have

$$A \cap \mathcal{G}(A) = A \setminus \mathcal{E}(A) = A \cap \mathcal{O}(A) \setminus \mathcal{E}(A) = \mathcal{O}(A) \setminus \mathcal{E}(A) = \mathcal{O}(A) \cap \mathcal{G}(A);$$

i.e. $A \cap \mathcal{G}(A)$ is relatively open in $\mathcal{G}(A)$. Similarly,

$$A \cap \mathcal{G}(A) = \mathcal{O}(A) \setminus \mathcal{E}(A) = \mathrm{cl}_Y(\mathcal{O}(A)) \setminus \mathcal{E}(A) = \mathrm{cl}_Y(\mathcal{O}(A)) \cap \mathcal{G}(A);$$

thus $A \cap \mathcal{G}(A)$ is relatively closed in $\mathcal{G}(A)$, as required.

We prove the lemma by induction on α. For $\alpha = 1$, B is a Borel set with open sections. So by Proposition 2.6, there are open sets $U(n) \subseteq X$ and Borel sets $B(n) \subseteq Y$ such that $B = \bigcup_{n<\omega}(U(n) \times B(n))$. Let $G = \bigcap_{n<\omega} \mathcal{G}(B(n))$. Since $B(n) \cap \mathcal{G}(B(n))$ is clopen in $\mathcal{G}(B(n))$, $B(n) \cap G$ is clopen in G. It follows that $(U(n) \times B(n)) \cap (X \times G)$ is open in $X \times G$. So $B \cap (X \times G) = \bigcup_{n<\omega}(U(n) \times B(n)) \cap (X \times G)$ is also open in $X \times G$. This completes the proof of the $\alpha = 1$ case.

Let now $1 < \alpha < \omega_1$ and suppose the statement holds for every $\beta < \alpha$. Let $B \subseteq X \times Y$ be a Borel set of which the section B^y is in $\mathbf{\Sigma}^0_\alpha(X)$ for every $y \in Y$. By Proposition 2.6, there are Borel sets $B(n) \subseteq X \times Y$ and ordinals $\alpha_n < \alpha$ such that $B = \bigcup_{n<\omega} B(n)$ and for every $n < \omega$ and every $y \in Y$ the section $(B(n))^y$ is in $\mathbf{\Pi}^0_{\alpha_n}(X)$. By applying the induction hypothesis to $Y \setminus B(n)$ $(n < \omega)$, we have comeager $\mathbf{\Pi}^0_2(Y)$ sets $G(n)$ such that $B(n) \cap (X \times G(n))$ is in $\mathbf{\Pi}^0_{\alpha_n}(X \times G(n))$. Then $G = \bigcap_{n<\omega} G(n)$ fulfills the requirements. □

An important corollary of Lemma 4.2 is the next lemma.

Lemma 4.3. *Let X and Y be Polish spaces and α be a countable nonzero ordinal. Let $B \subseteq X \times Y$ be a Borel set such that the horizontal section B^y is in $\mathbf{\Sigma}^0_\alpha(X)$ for every $y \in Y$. Then $\exists^{*Y} B$ is a $\mathbf{\Sigma}^0_\alpha(X)$ set.*

Proof. By Lemma 4.2 there is a comeager $\mathbf{\Pi}^0_2(Y)$ set G such that $B \cap (X \times G)$ is in $\mathbf{\Sigma}^0_\alpha(X \times G)$. We have $\exists^{*Y} B = \exists^{*G}(B \cap (X \times G))$ since G is comeager. By Lemma 2.5, the right hand side is a $\mathbf{\Sigma}^0_\alpha(X)$ set, as required. □

4.1 Proof of Theorem 1.4

For $\alpha = 0$, i.e., for continuous functions, the result is due to de Bruijn (see [2] or [14]). So we can assume $\alpha \geq 1$.

Let f satisfy the conditions of Theorem 1.4. By Lemma 4.1 there is a Baire class 1 function $g \colon \boldsymbol{R} \to \boldsymbol{R}$ such that $\{x \in \boldsymbol{R} \colon f(x) \neq g(x)\}$ is meager. Set $n = f - g$. Then it is enough to see that the Borel function n is of Baire class α. By [11, (24.3) Theorem p. 190], it is sufficient to show that for every open set $U \subseteq \boldsymbol{R}$ the inverse image $n^{-1}(U)$ is in $\mathbf{\Sigma}^0_{\alpha+1}(\boldsymbol{R})$.

We define $F \colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ by $F(x, y) = -n(x + y) + n(x) + n(y)$ and set $B = F^{-1}(U)$. For every $y \in \boldsymbol{R}$ the section $F^y = n(y) - \Delta_y n$ is of Baire class α. So by Lebesgue's theorem, $B^y = (F^y)^{-1}(U)$ is a $\mathbf{\Sigma}^0_{\alpha+1}(\boldsymbol{R})$ set. Then by Lemma 4.3 the set $\exists^{*\boldsymbol{R}} B$ is a $\mathbf{\Sigma}^0_{\alpha+1}(\boldsymbol{R})$ set.

Fix an arbitrary $x \in \boldsymbol{R}$. By definition,

$$B_x = \{y \in \boldsymbol{R} \colon -n(x + y) + n(x) + n(y) \in U\}.$$

Since both $\{y\colon n(x+y)=0\}$ and $\{y\colon n(y)=0\}$ are comeager, B_x is nonmeager if and only if $n(x)\in U$. Hence $n^{-1}(U)=\exists^{*\boldsymbol{R}}B$ which is a $\boldsymbol{\Sigma}^0_{\alpha+1}(\boldsymbol{R})$ set. This completes the proof.

To conclude this section, let us briefly indicate how Theorem 1.4 can be proved using a measure theoretic approach. The counterpart of Lemma 4.1 is the following result, which follows by a straightforward application of Luzin's Theorem (see e.g. [11, (17.12) Theorem p. 108]).

Lemma 4.4. *Let (X,τ) be a Polish space and let μ be a Borel measure on X satisfying $\mu(B(x,r))<\infty$ for every $x\in X$ and $r>0$. Let $f\colon X\to\boldsymbol{R}$ be a μ-measurable function. Then for every $\varepsilon>0$ there is a closed set $F\subseteq X$ and a continuous function $g\colon X\to\boldsymbol{R}$ such that $\mu(X\setminus F)<\varepsilon$ and $f|_F=g|_F$.*

We note that it is *not* enough to assume that μ is σ-finite and we cannot achieve $\mu(X\setminus F)=0$ even if we relaxed the condition on g to be merely of Baire class 1.

The counterpart of Lemma 4.2 is the following result.

Lemma 4.5. *Let X and Y be Polish spaces and let μ be a Borel measure on Y satisfying $\mu(B(y,r))<\infty$ $(y\in Y,\ r>0)$. Let α be a countable nonzero ordinal. Let $B\subseteq X\times Y$ be a Borel set such that the horizontal section B^y is in $\boldsymbol{\Sigma}^0_\alpha(X)$ for every $y\in Y$. Then for every $\varepsilon>0$ there is a closed set $F\subseteq Y$ such that $\mu(Y\setminus F)<\varepsilon$ and $B\cap(X\times F)$ is in $\boldsymbol{\Sigma}^0_\alpha(X\times F)$.*

Proof. By [11, (17.11) Theorem p. 107], for every Borel set $A\subseteq Y$ we have $\mu(A)=\sup\{\mu(K)\colon K\subseteq A,\ K \text{ closed}\}$. So for every Borel set $A\subseteq Y$ and every $\varepsilon>0$ there is a closed set $\mathcal{G}(A)\subseteq Y$ such that $\mu(Y\setminus\mathcal{G}(A))<\varepsilon$ and $A\cap\mathcal{G}(A)$ is clopen in $\mathcal{G}(A)$. Then the rest of the proof is an inductive argument, as for Lemma 4.2. □

The corollary of Lemma 4.5, analogous to Lemma 4.3, follows from a result of Montgomery (see e.g. [11, (22.25) Exercise p. 175]).

Lemma 4.6. *Let X and Y be Polish spaces and let μ be a Borel measure on Y satisfying $\mu(B(y,r))<\infty$ $(y\in Y,\ r>0)$. Let α be a countable nonzero ordinal and let $B\subseteq X\times Y$ be a Borel set such that the horizontal section B^y is in $\boldsymbol{\Sigma}^0_\alpha(X)$ for every $y\in Y$. Then for every $\varepsilon\geq 0$, $\{x\in X\colon\mu(B_x)>\varepsilon\}$ is a $\boldsymbol{\Sigma}^0_\alpha(X)$ set.*

Using these lemmas, the proof of Theorem 1.4 using Lebesgue measurability of Borel functions follows as in the Baire category approach above.

5 Generalizations

5.1 Difference property of Lebesgue measurable functions

Recall that $\mathrm{cov}(\mathcal{N})$ denotes the smallest cardinality of a family of Lebesgue null sets that covers $\boldsymbol{R}$; $\mathrm{non}(\mathcal{N})$ is the smallest cardinality of a Lebesgue non-null set; and $\mathrm{non}^\star(\mathcal{N})$ is the smallest cardinality κ such that every Lebesgue non-null set has a Lebesgue non-null subset of cardinality κ (see [1] for more on cardinal invariants).

It was shown in [13] that the difference property of Lebesgue measurable functions follows from the cardinal inequality $\mathrm{non}^\star(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$. This inequality is known to hold in random real models (see [13]). If κ is an atomlessly measurable cardinal, then we know $\mathrm{cov}(\mathcal{N}) = \kappa$ and $\mathrm{non}(\mathcal{N}) = \omega_1$. But we do not know the answer to the following.

Question *Is* $\mathrm{non}^\star(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$ *a consequence of the existence of an atomlessly measurable cardinal?*

We note here that, unlike $\mathrm{non}(\mathcal{N})$, the cardinal $\mathrm{non}^\star(\mathcal{N})$ is not determined by the presence of an atomlessly measurable cardinal. If κ is two-valued measurable and CH holds, then the forcing notion $\mathbb{B}_\kappa$ for adding κ many random reals forces that κ is atomlessly measurable and $\mathrm{non}^\star(\mathcal{N}) = \omega_1$. On the other hand, if κ is two-valued measurable and MA+$\neg$CH holds, then $\mathbb{B}_\kappa$ forces that κ is atomlessly measurable and $\mathrm{non}^\star(\mathcal{N}) > \omega_1$. However, as we have mentioned, $\mathrm{cov}(\mathcal{N})$ is forced to be not less than κ. Therefore it anyway becomes far bigger than $\mathrm{non}^\star(\mathcal{N})$ which does not exceed the size of ground model's continuum. So these two cases do not provide counter-example to our Question.

Finally, we would like to point out that Lemma 3.2 can be applied to a problem studied in [15], from which we have adopted Lemma 3.1. The modified version of [15, Theorem 2] is the following.

Proposition 5.1. *Suppose there exists an atomlessly measurable cardinal. Let* $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ *be such that the vertical section* f_x *is approximately continuous and the horizontal section* f_y *is Borel for every* $x, y \in \boldsymbol{R}$. *Then* f *is Borel as a function of two variables.*

5.2 Difference property of the Baire classes

Let us recall that by a classical result of N. G. de Bruijn, the class of continuous functions has the difference property, while by a result M. Laczkovich, the class of Baire class 1 functions has the difference property (see [2] and

[14]); that is, for $\alpha = 0$ and $\alpha = 1$ the conclusion of Theorem 1.4 holds without any definability assumption on f.

On the other hand, under $V = L$, there exists an analytic set $A \subseteq \boldsymbol{R}$ such that both A and $\boldsymbol{R} \setminus A$ are uncountable while $(A+t) \setminus A$ is countable for every $t \in \boldsymbol{R}$ (see [3, Theorem 4.7].) It is known that such an analytic set A is non-Borel and comeager, thus χ_A shows that the conclusion of Theorem 1.4 may fail for $\alpha = 2$ and a $\boldsymbol{\Delta}^1_2$-measurable function.

The main lemma of Theorem 1.4, Lemma 4.2 is also optimal in the following sense.

Proposition 5.2. *Assume $V = L$. Then there is a function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ of coanalytic graph such that Graph$(f) \cap (B \times \boldsymbol{R})$ is not Borel for every set B which is either non-null or non-meager.*

Proof. It is well known that under $V = L$ there exists a $\boldsymbol{\Delta}^1_2$ set which is not Lebesgue measurable in any non-null Borel set and does not have the Baire property in any non-meager Borel set; e.g., the well-ordering $<_L \cap ({}^\omega\omega \times {}^\omega\omega)$ (see e.g. [9, Corollary 25.28 p. 495]. So let $P \subseteq \boldsymbol{R}$ be such a $\boldsymbol{\Delta}^1_2$ set. Then both P and $\boldsymbol{R} \setminus P$ are projections of $\boldsymbol{\Pi}^1_1$ sets, i.e. there are $\boldsymbol{\Pi}^1_1$ sets $A_1 \subseteq \boldsymbol{R} \times [0, 1)$ and $A_2 \subseteq \boldsymbol{R} \times [1, 2)$ such that $P = \mathrm{proj}_{\boldsymbol{R}}(A_1)$ and $\boldsymbol{R} \setminus P = \mathrm{proj}_{\boldsymbol{R}}(A_2)$.

Let $A = A_1 \cup A_2$. By [11, (36.15) Theorem p. 306], A can be uniformized by a $\boldsymbol{\Pi}^1_1$ set, so we can assume A is the graph of a function $f\colon \boldsymbol{R} \to [0, 2)$. Therefore for every $x \in \boldsymbol{R}$ the vertical section A_x consists of a unique point, in particular it is closed.

Let $B \subseteq \boldsymbol{R}$ be a set satisfying $A \cap (B \times [0, 2))$ is Borel. Then $P_1 = \mathrm{proj}_{\boldsymbol{R}}\Big(A \cap (B \times [0, 1))\Big)$ and $P_2 = \mathrm{proj}_{\boldsymbol{R}}\Big(A \cap (B \times [1, 2))\Big)$ form a disjoint pair of analytic sets such that $P_1 \cup P_2 = B$ and $B \cap P = P_1$; in particular B and $B \cap P$ are both analytic. So if B is non-null then there is a non-null Borel set $B' \subseteq B$ such that $B' \cap P$ is Lebesgue measurable, a contradiction. Similarly, if B is non-meager then there is a non-meager Borel set $B' \subseteq B$ such that $B' \cap P$ has the Baire property, again a contradiction. This completes the proof. □

For the sake of completeness, we point out that for $\alpha = 1$, Lemma 4.2 *holds* for coanalytic sets.

Proposition 5.3. *Let X and Y be Polish spaces and let $A \subseteq X \times Y$ be a coanalytic set. Suppose for every $y \in Y$ the horizontal section A^y is open. Then there exists a comeager $\boldsymbol{\Pi}^0_2(Y)$ set G such that $A \cap (X \times G)$ is open in $X \times G$.*

Proof. For each basic open set I from a fixed countable open basis of X, put $B_I = \{y \in Y : I \subseteq A^y\}$. It is routine to check that B_I is coanalytic, in particular it has the Baire property. Then we can proceed as in the proof of Lemma 4.2. $\square$

Finally we would like to show that, e.g. under sufficient determinacy assumptions, Theorem 1.4 holds for functions satisfying weaker measurability assumptions, as follows. We call a pointclass $\mathbf{\Gamma}$ *adequate* if it is an algebra which contains the Borel sets and it is closed under taking Cartesian products.

Theorem 5.4. *Let $\mathbf{\Gamma}$ be an adequate pointclass such that the complements of projections of sets in $\mathbf{\Gamma}$ can be uniformized by Baire measurable functions. Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a $\mathbf{\Gamma}$-measurable function and let α be a countable ordinal. Suppose that for every $h \in \boldsymbol{R}$ the difference function $\Delta_h f$ is of Baire class α. Then f itself is of Baire class α.*

The assumption of Theorem 5.4 holds for $\mathbf{\Gamma} = \mathbf{\Delta}^1_n$ $(n < \omega)$ under Projective Determinacy (see e.g. [11, Chapter 39 p. 327] or [9, Chapter 33 p. 627]). Below we only state and prove the key ingredient of its proof, namely the suitable generalization of Lemma 4.2; then the statement follows as for Borel measurable functions.

Lemma 5.5. *Let X and Y be Polish spaces and let $\mathbf{\Gamma}$ be as in Theorem 5.4. Let α be a countable nonzero ordinal. Let $B \subseteq X \times Y$ be in $\mathbf{\Gamma}$ such that the horizontal section B^y is in $\mathbf{\Sigma}^0_\alpha(X)$ for every $y \in Y$. Then there is a comeager $\mathbf{\Pi}^0_2(Y)$ set G such that $B \cap (X \times G)$ is in $\mathbf{\Sigma}^0_\alpha(X \times G)$.*

Proof. Let $U \subseteq 2^\omega \times X$ be a universal $\mathbf{\Sigma}^0_\alpha$ set, i.e. such that for every $A \in \mathbf{\Sigma}^0_\alpha(X)$ there is a $t \in 2^\omega$ with $U_t = A$. Let

$$P = \{(y,t,x) \in Y \times 2^\omega \times X : \\ ((x,y) \in B \wedge (t,x) \notin U) \vee ((x,y) \notin B \wedge (t,x) \in U)\}.$$

Since $\mathbf{\Gamma}$ is an algebra which contains the Borel sets and is closed under taking Cartesian products, we have $P \in \mathbf{\Gamma}(Y \times 2^\omega \times X)$.

Note that $(y,t) \in \mathrm{proj}_{Y\times 2^\omega}(P)$ if and only if $B^y \neq U_t$. So

$$\{(y,t) \in Y \times 2^\omega : B^y = U_t\} = (Y \times 2^\omega) \setminus \mathrm{proj}_{Y\times 2^\omega}(P). \tag{1}$$

By our assumption on B, for every $y \in Y$ there exists a $t \in 2^\omega$ such that $B^y = U_t$. By our assumption on $\mathbf{\Gamma}$, there is a Baire measurable uniformizing function of the set in (1); i.e. there is a Baire measurable function $f\colon Y \to 2^\omega$ such that $B^y = U_{f(y)}$ $(y \in Y)$.

By Nikodým's Theorem (see e.g. [11, (8.38) Theorem p. 52]), there is a comeager $\mathbf{\Pi}^0_2(Y)$ set G such that $f|_G\colon G \to 2^\omega$ is continuous. We have $(x,y) \in B$ if and only if $(f(y),x) \in U$, so

$$B \cap (X \times G) = \{(x,y) \in X \times G\colon ((f|_G)(y),x) \in U\}.$$

Since $f|_G$ is continuous, this shows $B \cap (X \times G)$ is $\mathbf{\Sigma}^0_\alpha(X \times G)$, as required. □

Similar generalization is possible for Lemma 4.5 of the approach using Lebesgue measure. However, we are far from understanding the precise consistency strength of the difference property of the class of Borel measurable functions or of the class of Baire class α functions. So we do not present here any further analysis of the proofs.

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