Freiling’s Axiom of Symmetry ($A_{\aleph_0}$) is the following statement: For every function $F : 2^\omega \to [2^\omega]^{\leq \omega}$ which assigns a countable set of reals to each real, there exist two distinct reals, say $a$ and $b$, such that $a \notin F(b)$ and $b \notin F(a)$.

**Fact 1** (Freiling[1]). ZFC $\vdash A_{\aleph_0} \iff \neg \text{CH}$. $\triangleright$

Galen Weitkamp has considered (in [3]) an effective version of $A_{\aleph_0}$.

Fix a recursive bijection $\langle , \rangle : \omega \times \omega \to \omega$. For each $a \in 2^\omega$ and $n \in \omega$, the real $(a)_n \in 2^\omega$ is defined by $(a)_n(k) = a(\langle n, k \rangle)$. In this way every real $a \in 2^\omega$ naturally codes a countable set $\{ (a)_n : n \in \omega \}$.

**Definition.** Let $\Gamma$ be a pointclass. Then $A(\Gamma)$ states: Let $f : 2^\omega \to 2^\omega$ be a function whose graph as subset of $2^\omega \times 2^\omega$ belongs to the class $\Gamma$, then there exist two distinct reals $a$ and $b$ such that

$$\forall n \in \omega \left[ x \neq (f(y))_n \land y \neq (f(x))_n \right].$$

**Fact 2** (Weitkamp [3]).

(1) ZF + DC $\vdash A(\Sigma^1_1)$.

(2) $A(\Pi^1_1) \iff A(\Sigma^1_2) \iff 2^\omega \not\subset L$. $\triangleright$

Fact 2 (2) gives an effective version of Freiling’s Fact 1. However, there are some difficulties within Weitkamp’s formulation:

1. Freiling has considered $A_{\text{null}}$ and $A_{\text{meager}}$ as well, replacing “countable” by “null” and “meager” respectively. It is not clear how we can modify Weitkamp’s setting to handle these generalizations.

2. Giving a countable set of reals is not the same thing as giving its code. From a code you can easily obtain a countable set as Weitkamp does. But for each countable set $C \in [2^\omega]^{\leq \omega}$ there exist uncountably many reals which codes $C$, and you do not know how to choose one.

To investigate this second point more closely, suppose we are given a relation $R \subset 2^\omega \times 2^\omega$ which is somehow *nicely definable* (Borel, analytic, or
anything). Suppose also that for every \( x \in 2^\omega \) the vertical section \( R_x = \{ y : R(x, y) \} \) is nonempty and countable. In such a case can you always define a function \( f : 2^\omega \to 2^\omega \) such that \( R_x = \{ (f(x))_n : n \in \omega \} \)? For example, the following question should be a challenging exercise:

**Question 3.** Define a function \( f : 2^\omega \to 2^\omega \) so that

\[
\{ (f(x))_n : n \in \omega \} = \{ y \in 2^\omega : y \text{ is recursive in } x \}
\]

for every \( x \in 2^\omega \). At which level of the arithmetical hierarchy can such \( f \) be?

From this point of view, the following reformulation seems more natural to me.

**Definition.** Let \( A^*(\Gamma) \) state: For a relation \( R \subseteq 2^\omega \times 2^\omega \) in \( \Gamma \), if every vertical section \( R_x \) is countable, then there are two distinct reals \( a \) and \( b \) such that both \( R(a, b) \) and \( R(b, a) \) fail.

This is not always equivalent to Weitkamp’s \( A(\Gamma) \). We still have

\[
A^*(\Sigma^1_2) \leftrightarrow A^*(\Delta^1_2) \leftrightarrow 2^\omega \not\subseteq L,
\]

so \( A^*(\Sigma^1_2) \) and \( A(\Sigma^1_2) \) are equivalent. On the other hand, we have (by the Fubini Theorem)

\[
ZF + DC \models A^*(\Pi^1_1).
\]

Therefore \( A^*(\Pi^1_1) \) is strictly weaker than \( A(\Pi^1_1) \).

Our version has one obvious advantage. It is quite easy to formulate \( A^*_{\text{null}}(\Gamma) \) and \( A^*_{\text{meager}}(\Gamma) \). Then by Fubini and Kuratowski-Ulam Theorems,

**Fact 4.** For every pointclass \( \Gamma \),

1. \( \text{LM}(\Gamma) \rightarrow A^*_{\text{null}}(\Gamma) \), and
2. \( \text{BP}(\Gamma) \rightarrow A^*_{\text{meager}}(\Gamma) \). \( \triangleleft \)

It is amusing to point out that in certain cases these arrows are inverted.

**Fact 5.**

1. \( \text{LM}(\Delta^1_2) \leftrightarrow A^*_{\text{null}}(\Delta^1_2) \), and
2. \( \text{BP}(\Delta^1_2) \leftrightarrow A^*_{\text{meager}}(\Delta^1_2) \).
Here, I will give only a proof of (1), since (2) can be proved similarly.

We already know that $\text{LM}(\Delta^1_1)$ implies $A^*_{\text{null}}(\Delta^1_1)$. To see the converse, suppose that $\text{LM}(\Delta^1_1)$ fails. Then there is no random real over $L$. In other words, every real $r \in 2^{\omega}$ belongs to some null $G_\delta$ set with constructible code.

Let $U \subset 2^{\omega} \times 2^{\omega}$ be a universal $G_\delta$ set which is lightface $\Pi^0_2$. Then our hypothesis $\neg \text{LM}(\Delta^1_1)$ can be written as

$$\forall r \in 2^{\omega} \exists c \in 2^{\omega} \left[ c \in L \land \mu(U_c) = 0 \land r \in U_c \right].$$

where $\mu$ denotes the Lebesgue measure. Since the $\left[ \ldots \right]$ part of the statement is $\Sigma^1_2$, the Novikov-Kondô-Addison Theorem gives a $\Delta^1_2$ function $\varphi : 2^{\omega} \to 2^{\omega}$ such that

$$\forall r \in 2^{\omega} \left[ \varphi(r) \in L \land \mu(U_{\varphi(r)}) = 0 \land r \in U_{\varphi(r)} \right].$$

Let $<^*$ be a $\Sigma^1_2$ wellordering of $2^{\omega} \cap L$ into order-type $\omega_1$. We may assume

$$L \models \left[ <^* \text{ is a } \Sigma^1_2 \text{-good wellordering} \right]$$

in the sense explained in Section 5A of [2]. Now define $R \subset 2^{\omega} \times 2^{\omega}$ by

$$R(x, y) \iff \exists c \leq^* \varphi(x) \left[ \mu(U_c) = 0 \land y \in U_c \right].$$

It is straightforward to see that every vertical section $R_x$ is null and that every two reals $a$ and $b$ satisfy either $R(a, b)$ or $R(b, a)$ according to $\varphi(b) \leq^* \varphi(a)$ or not. Thus what remains to see is:

**Lemma 6.** The relation $R$ is $\Delta^1_2$.

**Proof.** Let $\text{IS}(x, y)$ be the predicate that tells $x$ codes the initial segment of $\leq^*$ with top $y$. Exercise 5A.1 of [2] shows that $V = L$ implies that $\text{IS}$ is $\Delta^1_2$. Even when $V \neq L$, the predicate

$$\text{IS}'(x, y) \iff x, y \in 2^{\omega} \cap L \land L \models \text{IS}(x, y)$$

is still $\Sigma^1_2$. We then have

$$\neg R(x, y) \iff \forall c \leq^* \varphi(x) \left[ \mu(U_c) > 0 \lor y \notin U_c \right] \iff \exists b \left[ b \in L \land \text{IS}'(b, \varphi(x)) \land \forall n \in \omega \left[ \mu(U_{(b), n}) > 0 \lor y \notin U_{(b), n} \right] \right]$$

which gives a $\Sigma^1_2$ description of negation of $R$. \qedsymbol

This completes the proof of Fact 5.

**Question 7.** Does $A^*_{\text{null}}(\Sigma^1_2)$ imply $\text{LM}(\Sigma^1_2)$?
References

