QUASI-CONVEXLY DENSE AND SUITABLE SETS IN THE ARC COMPONENT OF A COMPACT GROUP

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Dedicated to Karl H. Hofmann on the occasion of his 76th anniversary

Abstract. Let G be an abelian topological group. The symbol $\widehat{G}$ denotes the group of all continuous characters $\chi : G \to \mathbb{T}$ endowed with the compact open topology. A subset E of G is said to be *qc-dense in* G provided that $\chi(E) \subseteq \varphi([-1/4, 1/4])$ holds only for the trivial character $\chi \in \widehat{G}$, where $\varphi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the canonical homomorphism. A *super-sequence* is a non-empty compact Hausdorff space S with at most one non-isolated point (to which S *converges*). We prove that an infinite compact abelian group G is connected if and only if its arc component G_a contains a super-sequence converging to 0 that is qc-dense in G. This gives as a corollary a recent theorem of Außenhofer: For a connected locally compact abelian group G, the restriction homomorphism $r : \widehat{G} \to \widehat{G}_a$ defined by $r(\chi) = \chi \restriction_{G_a}$ for $\chi \in \widehat{G}$, is a topological isomorphism. We show that an infinite compact group G is connected if and only if its arc component G_a contains a super-sequence converging to the identity that is qc-dense in G and generates a dense subgroup of G. We also offer a short alternative proof of the result of Hofmann and Morris on the existence of suitable sets of minimal size in the arc component of a compact connected group [15, Theorem 12.42].

1. Introduction

The symbol $w(X)$ denotes the *weight* of a topological space X, $\mathfrak{c}$ denotes the cardinality of the continuum and $\mathbb{N}$ denotes the set of natural numbers. *All topological groups are assumed to be Hausdorff.*

Let G be a topological group. We denote by $\widehat{G}$ the group of all continuous characters $\chi : G \to \mathbb{T}$ endowed with the compact open topology. A subgroup D of G *determines* G if the restriction homomorphism $r : \widehat{G} \to \widehat{D}$ defined by $r(\chi) = \chi \restriction_D$ for $\chi \in \widehat{G}$, is a topological isomorphism [5]. If G is locally compact and abelian, then every subgroup D that determines G must be dense in G. Furthermore, when D is dense in G, the map $r : \widehat{G} \to \widehat{D}$ is a continuous isomorphism.

Let us recall three cornerstone results in the topic of determining subgroups.

Theorem 1.1. [1, 4] *A metrizable abelian group G is determined by each dense subgroup of G.*

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Theorem 1.2. ([13]; proved earlier in [5] under the assumption of CH) *Every non-metrizable compact abelian group G contains a dense subgroup that does not determine G.*

Theorem 1.3. [3] *The arc component G_a of a connected locally compact abelian group G determines G.*

Obviously, Theorem 1.2 inverts Theorem 1.1 for compact abelian groups.

Theorem 1.3 is used in [2] to prove that the uncountable powers of $\mathbb{Z}$ are not strongly reflexive, thereby resolving a problem raised by Banaszczyk on whether uncountable powers of the reals $\mathbb{R}$ are strongly reflexive.

According to a well-known result of Eilenberg and Pontryagin, in a connected locally compact abelian group G the arc component G_a is dense. Since a subgroup of a locally compact abelian group determining it must be dense, Theorem 1.3 is a strengthening of this classical result.

While Theorem 1.3 is a corollary of Theorem 1.1 for a metrizable group G, in the non-metrizable case the mere density of G_a in G (ensured by the classical result of Eilenberg and Pontryagin) need not guarantee that G_a determines G, as witnessed by Theorem 1.2.

Definition 1.4. Let $\varphi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the canonical homomorphism and $\mathbb{T}_+ = \varphi([-1/4, 1/4])$. We will say that a non-empty subset E of a topological group G is *qc-dense in G* (an abbreviation for *quasi-convexly dense*) provided that $\chi(E) \subseteq \mathbb{T}_+$ only for the trivial continuous homomorphism $\chi : G \to \mathbb{T}$.

This notion was introduced in [6] in the abelian context, and its significance for applications has been recently demonstrated in [10]. In particular, qc-density was used in [10] to establish essential properties of determining subgroups of compact abelian groups, thereby allowing to get a short elementary proof of Theorem 1.2.

The host of applications of qc-dense sets is made possible by the ultimate connection between the notions of determining subgroup and qc-density described in the next fact:

Fact 1.5. *A subgroup D of a compact abelian group G determines it if and only if there exists a compact subset of D that is qc-dense in G.*

Fact 1.5 is proved in [10, Fact 1.4]. It is a particular case of a more general fact stated without proof (and in equivalent terms) in [5, Remark 1.2(a)] and [13, Corollary 2.2].

It has been recently shown in [10] that qc-dense compact subsets (and thus determining subgroups) of a compact abelian group must be rather big.

Theorem 1.6. [10, Corollary 2.2] *If a closed subset X of an infinite compact abelian group G is qc-dense in G, then $w(X) = w(G)$.*

A *super-sequence* is a non-empty compact Hausdorff space X with at most one non-isolated point x^* [9]. We will call x^* the *limit* of X and say that X *converges to* x^*. Observe that a countably infinite super-sequence is a convergent sequence (together with its limit).

Being an immediate consequence of [1, Theorem 4.3 or Corollary 4.4], the following result is essentially due to Außenhofer:

Theorem 1.7. [1] *Every dense subgroup D of an infinite compact metric abelian group G contains a sequence converging to 0 that is qc-dense in G.*

In particular, every infinite compact metric abelian group has a qc-dense sequence converging to 0. This statement has been recently extended to all compact groups by replacing convergent sequences with super-sequences:

Theorem 1.8. [10] *Every infinite compact abelian group contains a qc-dense super-sequence converging to* 0.

A common strengthening of both Theorems 1.7 and 1.8 is impossible. Indeed, every non-metrizable compact abelian group G contains a dense subgroup H such that no super-sequence $S \subseteq H$ is qc-dense in G. To see this, apply Theorem 1.2 to get a dense subgroup H of G that does not determine G, and then notice that any super-sequence $S \subseteq H$ (being compact) cannot be qc-dense in G by Fact 1.5.

Definition 1.9. Let G be a topological group with the identity e.

- (i) A subspace X of G *topologically generates* G if the subgroup of G generated by X is dense in G.
- (ii) If a discrete subset S of G topologically generates G and $S \cup \{e\}$ is closed in G, then S is called a *suitable set for* G [14].

The proof of the following fact is straightforward.

Fact 1.10. [10, Fact 1.3(ii)] *Every qc-dense subset of a compact abelian group topologically generates it.*

A *convergent* super-sequence is never a suitable set and a suitable set is never a *convergent* super-sequence. Nevertheless, there is a close relation between these two concepts summarized in the following

Remark 1.11. Clearly, if S is a super-sequence in G that converges to e and topologically generates G, then $S \setminus \{e\}$ is a suitable set for G. Conversely, if G is compact and S is a suitable set for G, then $S \cup \{e\}$ must be a super-sequence.

It follows from Remark 1.11 that a subgroup D of a compact group G contains a super-sequence converging to the identity that topologically generates G if and only if D contains an infinite suitable set for G. In the "if" part of this statement the assumption that the suitable set is infinite is essential. Indeed, there exists a dense monothetic subgroup D of the compact abelian group $G = \mathbb{T}^{\mathfrak{c}}$ such that all compact subsets of D are finite [12]. (Note that a monothetic group has a suitable set consisting of a singleton.)

Remark 1.12.
- (i) *If a super-sequence S converging to 0 is qc-dense in a compact abelian group G, then $S \setminus \{0\}$ is a suitable set for G.* This is an immediate corollary of Fact 1.10 and Remark 1.11.
- (ii) *A suitable set for a compact abelian group G need not be qc-dense in G.* Indeed, it is well-known that the group $\mathbb{T}^{\mathfrak{c}}$ is monothetic, that is, topologically generated by a singleton S. Clearly, S is a suitable set for $\mathbb{T}^{\mathfrak{c}}$. Since $w(S) \leq \omega < \mathfrak{c} = w(\mathbb{T}^{\mathfrak{c}})$, S cannot be qc-dense in $\mathbb{T}^{\mathfrak{c}}$ by Theorem 1.6.

Hofmann and Morris discovered the following fundamental result:

Theorem 1.13. ([14]; see also [15]) *Every locally compact group has a suitable set.* [1]

Theorem 1.8 implies the particular case of Theorem 1.13 for compact abelian groups. Indeed, let G be a compact abelian group. If G is finite, then $G \setminus \{0\}$ is obviously a suitable set for G. If G is infinite, then Theorem 1.8 guarantees the existence of a qc-dense super-sequence S in G converging to 0. Now $S \setminus \{0\}$ is a suitable set for G by Remark 1.12(i). In

[1] A "purely topological" proof of this result based on Michael's selection theorem can be found in [17].

the opposite direction, it follows from Remark 1.12(ii) that the particular case of Theorem 1.13 for compact abelian groups does not imply Theorem 1.8.

Theorem 1.13 allowed Hofmann and Morris to introduce the *generating rank*

$$s(G) = \min\{|S| : S \text{ is a suitable set for } G\}$$

of a (locally) compact group G (see [15, Definition 12.15(i)]). This is undoubtedly one of the most important cardinal invariants of a compact group, as witnessed by the fact that the last chapter of the monograph [15] by Hofmann and Morris is entirely devoted to the study of this cardinal function and its relation to the weight. In particular, a complete computation of $s(G)$ in terms of $w(G)$ for a compact group G has been obtained in [15, Section 12] (see also [9] for alternative self-contained proofs). We will state explicitly only a particular case that we will need in this manuscript.

For a cardinal number κ define $\sqrt[\omega]{\kappa} = \min\{\tau \geq \omega : \tau^\omega \geq \kappa\}$.

Theorem 1.14. ([15, Theorem 12.25]; see also [9, Corollary 9.2]) *Let G be a compact connected group. Then:*

(i) *$s(G) \leq \sqrt[\omega]{w(G)}$;*
(ii) *if $w(G) > \mathfrak{c}$, then $s(G) = \sqrt[\omega]{w(G)}$.*

The concluding theorem from the monograph [15] establishes the existence of suitable sets of minimal size in the arc component of a compact connected group:

Theorem 1.15. [15, Theorem 12.42] *The arc component G_a of a compact connected group G contains a suitable set S for G such that $|S| = s(G)$.*

In Section 5 we apply the techniques developed in this paper and the author's manuscript [9] to offer a short alternative proof of this theorem.

2. Results

Our first result characterizes connected compact abelian groups in the spirit of Theorem 1.8.

Theorem 2.1. *For an infinite compact abelian group G the following conditions are equivalent:*

(i) *the arc component G_a of G contains a super-sequence of size $\leq w(G)$ converging to 0 that is qc-dense in G;*
(ii) *G_a contains a suitable set S for G such that $|S| = s(G)$;*
(iii) *G is connected.*

In view of Fact 1.5, the implication (iii)→(i) of Theorem 2.1 yields Theorem 1.3 when G is compact. The general case of Theorem 1.3 easily follows from the compact case, see the proof in the end of Section 4. As a by-product, we obtain an alternative, short and self-contained proof of Theorem 1.3.

The implication (iii)→(ii) of Theorem 2.1 coincides with the abelian case of Theorem 1.15 (that is, with the conjunction of Lemma 12.32 and Assertion (B) from [15]). It is this abelian case that presents the main difficulty in the proof of Theorem 1.15 (see [15, Lemma 12.31]).

Our second result characterizes connected compact groups in the spirit of Theorem 1.8. We denote by $c(Z(G))$ is the connected component of the center $Z(G)$ of G.

Theorem 2.2. *For an infinite compact group G the following conditions are equivalent:*

(i) *the arc component* G_a *of* G *contains a suitable set for* G*;*
(ii) G_a *contains an infinite suitable set for* G *that is qc-dense in* G*;*
(iii) G *is connected and* G_a *contains a super-sequence* S *that is qc-dense in* G*, topologically generates* $c(Z(G))$ *and satisfies the inequality* $|S| \leq w(c(Z(G)))$*; furthermore,* S *converges to* e_G *whenever* $c(Z(G))$ *is non-trivial;*
(iv) G *is connected.*

Remark 2.3. (a) Let $G = \mathbb{T}^{\mathfrak{c}}$. According to Remark 1.12(ii), $s(G) = 1$ yet no singleton is qc-dense in G. Since $G_a = G$, this shows that no suitable set S for G (inside G_a) such that $|S| = s(G)$ is qc-dense in G. We conclude that Theorem 1.15 does *not* imply Theorem 2.2.

(b) One *cannot* "merge" Theorems 1.15 and 2.2 by adding the following item to the list of equivalent conditions in Theorem 2.2:

(v) G_a contains a suitable set S for G that is qc-dense in G and satisfies $|S| = s(G)$.

Indeed, the example from item (a) shows that the implication (iv)→(v) fails.

(c) One *cannot* weaken item (iii) in Theorem 2.2 to the following one:

(ii*) G_a contains a super-sequence converging to the identity that is qc-dense in G.

Indeed, for every finite simple non-commutative group L the compact group $G = \mathbb{T} \times L$ has a sequence $S \subseteq G_a = \mathbb{T} \times \{e\}$ converging to the identity $(0, e)$ of G that is qc-dense in G, see Example 5.3. Since G is not connected, the implication (ii*)→(iv) fails.

Definition 2.4. Define the *qc-weight* $qcw(G)$ of a compact group G by

$$qcw(G) = \min\{|X| : X \text{ is a closed qc-dense subset of } G\}.$$

Using this new cardinal invariant, Theorem 1.6 can be restated as follows:

$$qcw(G) = w(G) \text{ for an infinite compact abelian group } G. \tag{1}$$

Our next theorem computes the value of $qcw(G)$ for an infinite compact connected group G.

Theorem 2.5. *Let* G *be an infinite compact connected group. Then* $qcw(G) = w(c(Z(G)))$.

With the help of this theorem we can now characterize compact connected groups which satisfy condition (v) from Remark 2.3.

Corollary 2.6. *For an infinite compact connected group* G *the following conditions are equivalent:*

(a) $s(G) \geq w(c(Z(G)))$*;*
(b) *there exists a suitable set for* G *is size* $s(G)$ *that is qc-dense in* G*;*
(c) G_a *contains a suitable set for* G *of size* $s(G)$ *that is qc-dense in* G.

Proof. The implication (c)→(b) is trivial, the implication (b)→(a) follows from Theorem 2.5. Let us prove the remaining implication (a)→(c). Applying Theorem 2.2, we can find a super-sequence S contained in G_a such that $|S| \leq w(c(Z(G)))$ and S is qc-dense in G. According to Theorem 1.15, there exists a suitable set T for G contained in G_a with $|T| = s(G)$. Now $X = S \cup T$ is a suitable set for G contained in G_a that is qc-dense in G and satisfies $|X| = \max\{s(G), w(c(Z(G)))\} = s(G)$. □

Combining the obvious inequality $s(G) \leq w(G)$ with (1), we conclude that $s(G) \leq qcw(G)$ for an infinite compact abelian group G. Our next example demonstrates that the cardinal invariants $qcw(G)$ and $s(G)$ become "independent of each other" when one drops the assumption of commutativity (even in the class of compact connected groups).

Example 2.7. Fix infinite cardinals κ and λ. Take a simple connected Lie group L_0 and define $K = \widehat{\mathbb{Q}}^\kappa$ and $L = L_0^\lambda$. Then $G = K \times L$ is an infinite compact connected group.

(a) $qcw(G) = \kappa$. Indeed, $c(Z(G)) = Z(G) = K \times \{e_L\}$, and so $w(c(Z(G))) = \kappa$. Now apply Theorem 2.5.

(b) If $\max\{\kappa, \lambda\} > \mathfrak{c}$, then $s(G) = \max\{\kappa_1, \lambda_1\}$, where $\kappa_1 = \sqrt[\omega]{\kappa}$ and $\lambda_1 = \sqrt[\omega]{\lambda}$ (Theorem 1.14).

(c) If $\lambda > 2^\kappa$, then $qcw(G) < s(G)$. Indeed, $qcw(G) = \kappa$ by (a), and $s(G) \geq \lambda_1 > \kappa$ by (b).

(d) If $\max\{\kappa, \lambda\} > \mathfrak{c}$, $\kappa_1 < \kappa$ and $\lambda < \kappa$, then $s(G) < qcw(G)$. Indeed, $\lambda_1 \leq \lambda < \kappa$ and $\kappa_1 < \kappa$ imply $s(G) = \max\{\kappa_1, \lambda_1\} < \kappa = qcw(G)$ by (b) and (a), respectively.

Problem 2.8. Compute $qcw(G)$ for any infinite compact group G.

Remark 2.9. *There exists a dense (connected, locally connected, countably compact) subgroup H of (the compact, connected abelian group) $G = \mathbb{T}^{2^\mathfrak{c}}$ such that H contains no suitable set for G.* Indeed, one can take as H the dense subgroup of G without a suitable set for H constructed in [11, Corollary 2.9].

The proofs of all theorems are postponed until Sections 4 and 5.

3. A qc-dense super-sequence in the arc component of $\widehat{\mathbb{Q}}$

Our main result in this section is Lemma 3.4. It follows from the density of $\widehat{\mathbb{Q}}_a$ in $\widehat{\mathbb{Q}}$ and Theorem 1.7. However, Außenhofer's proof of Theorem 1.7 relies on Arzela-Ascoli theorem and an inductive construction, so the qc-dense sequence she constructs in her proof is "generic". To keep this manuscript self-contained, we provide a "constructive" example of a "concrete" qc-dense sequence in $\mathbb{Q}_a$.

The proof of the following fact is straightforward from the definition.

Fact 3.1. *Let G and H be topological groups and $\pi : H \to G$ a continuous surjective group homomorphism. If a subset E of H is qc-dense in H, then $\pi(E)$ is qc-dense in G.*

Example 3.2. Let $T = \left\{\frac{1}{2n} : n \in \mathbb{N}, n \geq 1\right\} \cup \{0\}$. *The set $\varphi(T)$ is a qc-dense sequence in $\mathbb{T}$ converging to* 0. Indeed, let $\chi \in \widehat{\mathbb{T}}$ be a non-zero character. Then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $\chi(x) = mx$ for all $x \in \mathbb{T}$. Let $n = |m|$. Then $\frac{1}{2n} \in T$ and so $x = \varphi\left(\frac{1}{2n}\right) \in \varphi(T)$. Since $\chi(x) = mx = \varphi\left(\frac{m}{2n}\right) = \varphi\left(\frac{1}{2}\right) \notin \mathbb{T}_+$, we have $\chi(\varphi(T)) \setminus \mathbb{T}_+ \neq \emptyset$. This proves that $\varphi(T)$ is qc-dense in $\mathbb{T}$.

For $g \in G$ the symbol $\langle g \rangle$ denotes the cyclic subgroup of G generated by g. For a prime number p let $\mathbb{Z}_p$ denote the group of p-adic integers.

Lemma 3.3. *Let $\mathbb{P} = \{p_n : n \in \mathbb{N}\}$ be a faithful enumeration of the set $\mathbb{P}$ of prime numbers. Define*

$$H = \prod_{n \in \mathbb{N}} \mathbb{Z}_{p_n},$$

and let $v = \{1_{p_n}\}_{n \in \mathbb{N}} \in H$, where each 1_{p_n} is the identity of $\mathbb{Z}_{p_n}$. For $n \in \mathbb{N}$ define $k_n = (p_0 p_1 \dots p_{n-1})^n$. Then the set

$$S = \{m k_n v : n \in \mathbb{N}, m \leq k_{n+1}\} \cup \{0\} \subseteq \langle v \rangle \tag{2}$$

is a sequence converging to 0 that is qc-dense in H.

Proof. For $n \in \mathbb{N}$ define

$$W_n = k_n H = p_0^n \mathbb{Z}_{p_0} \times p_1^n \mathbb{Z}_{p_1} \times \ldots \times p_{n-1}^n \mathbb{Z}_{p_{n-1}} \times \prod_{i=n}^{\infty} \mathbb{Z}_{p_i}. \tag{3}$$

(Note that $k_0 = 1$, so $W_0 = H$.) Then $\{W_n : n \in \mathbb{N}\}$ forms a base of H at 0 consisting of clopen subgroups. It is easy to see that each W_n may miss only finitely many members of S, so S is a sequence converging to 0 in H.

Let us show that S is qc-dense in H. Let $\chi \in \widehat{H}$ and $\chi \neq 0$. We need to prove that $\chi(S) \setminus \mathbb{T}_+ \neq \emptyset$. Being a continuous homomorphic image of the compact totally disconnected group H, $\chi(H)$ is a closed totally disconnected subgroup of $\mathbb{T}$. Therefore, $\chi(H)$ must be finite. Hence $\ker \chi$ is an open subgroup of H, and consequently it contains a subgroup W_n for some $n \in \mathbb{N}$. Without loss of generality we will assume that

$$n = \min\{m \in \mathbb{N} : W_m \subseteq \ker \chi\}. \tag{4}$$

Since $\ker \chi \neq H = W_0$ by our assumption, we have $n \geq 1$, and so $n - 1 \in \mathbb{N}$.

Claim: $\chi(k_{n-1}v) \neq 0$.

Proof. Assume the contrary. Then $\chi\restriction_{\langle k_{n-1}v\rangle} = 0$. Since $\langle v\rangle$ is dense in H and W_{n-1} is an open subset of H, it follows that $\langle v\rangle \cap W_{n-1} = \langle k_{n-1}v\rangle$ is dense in W_{n-1}. Now from $\chi\restriction_{\langle k_{n-1}v\rangle} = 0$ and continuity of χ we conclude that $\chi\restriction_{W_{n-1}} = 0$. This gives $W_{n-1} \subseteq \ker \chi$, in contradiction with (4). □

Since $k_n v \in W_n \subseteq \ker \chi$ by (3) and (4), we have $k_n \chi(v) = \chi(k_n v) = 0$. That is, $\langle \chi(v)\rangle$ is a cyclic group of order at most k_n. Since $\chi(k_{n-1}v) = k_{n-1}\chi(v) \in \langle \chi(v)\rangle$, the order of the element $\chi(k_{n-1}v)$ of $\mathbb{T}$ is also at most k_n. Since $\chi(k_{n-1}v) \neq 0$ by claim, we can choose an integer $m \leq k_n$ such that $\chi(mk_{n-1}v) = m\chi(k_{n-1}v) \notin \mathbb{T}_+$. From (2) we conclude that $mk_{n-1}v \in S$, and so $\chi(S) \setminus \mathbb{T}_+ \neq \emptyset$. □

An explicit qc-dense sequence in $\widehat{\mathbb{Q}}$ converging to 0 can be found in [10, Lemma 4.7]. However, that sequence is not contained in $\widehat{\mathbb{Q}}_a$. In our next lemma we produce a qc-dense sequence converging to 0 *inside* $\widehat{\mathbb{Q}}_a$.

Lemma 3.4. *$\widehat{\mathbb{Q}}_a$ contains a sequence converging to 0 that is qc-dense in $\widehat{\mathbb{Q}}$.*

Proof. We continue using notations from Lemma 3.3. Let $K = \mathbb{R} \times H$ and $u = (1, v) \in K$. Then the cyclic subgroup $\langle u\rangle$ of K is discrete and the quotient group $C = K/\langle u\rangle$ is isomorphic to $\widehat{\mathbb{Q}}$ [7, §2.1]. Therefore, it suffices to prove that C_a contains a sequence converging to 0 that is qc-dense in C.

Let $\pi : K \to C = K/\langle u\rangle$ be the quotient homomorphism. Since π is a local homeomorphism, every continuous map $f : [0, 1] \to C$ with $f(0) = 0_C$ can be lifted to a continuous map $\widetilde{f} : [0, 1] \to K$ with $\widetilde{f}(0) = 0_K$ and $\pi \circ \widetilde{f} = f$. (A more general statement can be found in [16, Lemma 1].) Therefore, $C_a = \pi(K_a)$. Since H is zero-dimensional and $\mathbb{R} \times \{0\}$ is arcwise connected, one has $K_a = \mathbb{R} \times \{0\}$, and so $C_a = \pi(\mathbb{R} \times \{0\})$.

Define $N = \pi(\{0\} \times H)$, and let $f : C \to C/N$ be the quotient homomorphism. By Lemma 3.3 and Fact 3.1, there exists a converging to 0 sequence S' in the subgroup $\langle \pi(0, v)\rangle$ of N such that S' is qc-dense in N. As

$$\pi(0, v) = \pi((-1, 0) + (1, v)) = \pi(-1, 0) + \pi(u) = -\pi(1, 0) \in \pi(\mathbb{R} \times \{0\}) = C_a,$$

one has $S' \subseteq \pi(\langle(0,v)\rangle) \subseteq C_a$.

With $T = \left\{\frac{1}{2n} : n \in \mathbb{N}, n \geq 1\right\}$ define $S'' = \pi(T \times \{0\}) \subseteq \pi(\mathbb{R} \times \{0\}) = C_a$. Clearly, S'' is a sequence converging to 0. Since $C/N \cong K/(\mathbb{Z} \times H) \cong \mathbb{T}$ and the composed isomorphism $C/N \to \mathbb{T}$ sends $f(S'')$ to $\varphi(T)$, from Example 3.2 we conclude that $f(S'')$ is qc-dense in C/N.

Since S' and S'' are sequences converging to 0 in C, so is $X = S' \cup S''$. By our construction, $X \subseteq C_a$. So it remains only to prove that X is qc-dense in C. Suppose that $\chi \in \widehat{C}$ and $\chi(X) \subseteq \mathbb{T}_+$. Since $\chi \restriction_N \in \widehat{N}$, $\chi \restriction_N (S') = \chi(S') \subseteq \chi(X) \subseteq \mathbb{T}_+$ and S' is qc-dense in N, we have $\chi \restriction_N = 0$. Therefore, $\chi = \xi \circ f$ for some $\xi \in \widehat{C/N}$. In particular, $\xi(f(S'')) \subseteq \xi(f(X)) = \chi(X) \subseteq \mathbb{T}_+$. Since $f(S'')$ is qc-dense in C/N, it follows that $\xi = 0$. This gives $\chi = 0$. Therefore, X is qc-dense in C. $\square$

4. The abelian case: Proofs of Theorems 2.1 and 1.3

The following definition is an adaptation to the abelian case of [9, Definition 4.5]:

Definition 4.1. Let $\{G_i : i \in I\}$ be a family of abelian topological groups. For every $i \in I$ let X_i be a subset of G_i. Identifying each G_i with a subgroup of the direct product $G = \prod_{i \in I} G_i$ in the obvious way, define $X = \{0\} \cup \bigcup_{i \in I} X_i$, where 0 is the zero element of H. We will call X the *fan* of the family $\{X_i : i \in I\}$ and will denote it by $\mathrm{fan}_{i \in I}(X_i, G_i)$.

The proofs of the following two lemmas are straightforward.

Lemma 4.2. [10, Lemmas 4.3 and 4.4] *Let $\{G_i : i \in I\}$ be a family of abelian topological groups, and let $G = \prod_{i \in I} G_i$. For every $i \in I$ let X_i be a subset of G_i, and let $X = \mathrm{fan}_{i \in I}(X_i, G_i)$. Then:*

- (i) *if X_i is a sequence converging to 0 in G_i, then X is a super-sequence in G converging to 0;*
- (ii) *if X_i is a qc-dense subset of G_i for each $i \in I$, then X is qc-dense in G.*

Lemma 4.3. ([9, Fact 4.3]; see [17, Fact 12] for the proof) *Let $\pi : H \to G$ be a continuous group homomorphism and $S \subseteq H$. Then:*

- (i) *if S is a super-sequence, then so is $\pi(S)$;*
- (ii) *if S is a super-sequence converging to the identity e_H of H and $\pi(S)$ is infinite, then $\pi(S)$ is a super-sequence converging to the identity e_G of G.*

Lemma 4.4. *For every cardinal $\kappa > 0$ there exists a super-sequence $S \subseteq (\widehat{\mathbb{Q}}^\kappa)_a$ converging to 0 that is qc-dense in $\widehat{\mathbb{Q}}^\kappa$ and satisfies $|S| \leq \max\{\kappa, \omega\}$.*

Proof. Write $\widehat{\mathbb{Q}}^\kappa$ as $\widehat{\mathbb{Q}}^\kappa = \prod_{\alpha < \kappa} G_\alpha$, where G_α is the α's copy of $\widehat{\mathbb{Q}}$. By Lemma 3.4, for every $\alpha \in \kappa$ there is a sequence S_α in $(G_\alpha)_a$ converging to 0 that is qc-dense in G_α. By Lemma 4.2(i), $S = \mathrm{fan}_{\alpha \in \kappa}(X_\alpha, G_\alpha)$ is a super-sequence in $\widehat{\mathbb{Q}}^\kappa$ converging to 0. By Lemma 4.2(ii), S is qc-dense in $\widehat{\mathbb{Q}}^\kappa$. Finally, note that $S \subseteq \bigoplus_{\alpha \in \kappa} (G_\alpha)_a \subseteq (\widehat{\mathbb{Q}}^\kappa)_a$. $\square$

Proof of Theorem 2.1. Let $\kappa = w(G)$. Since G is infinite, $\kappa \geq \omega$. By [10, Theorem 3.3], there exits a continuous surjective homomorphism $\pi : \widehat{\mathbb{Q}}^\kappa \to G$.

(iii)$\to$(i) Let S be as in the conclusion of Lemma 4.4. Then $|\pi(S)| \leq |S| \leq w(G)$. Since S is qc-dense in $\widehat{\mathbb{Q}}^\kappa$, $\pi(S)$ is qc-dense in G by Fact 3.1. Since a finite set cannot be qc-dense in an infinite compact group ([1]; this also follows from Theorem 1.6), $\pi(S)$ must be infinite. By Lemma 4.3(ii), $\pi(S)$ is a super-sequence converging to 0. Finally note that $\pi(S) \subseteq \pi(\widehat{\mathbb{Q}}_a^\kappa) \subseteq G_a$.

(iii)→(ii) The connected metrizable group $\widehat{\mathbb{Q}}_a$ is monothetic, so applying [9, Corollary 5.9] and [9, Fact 4.4] we conclude that $(\widehat{\mathbb{Q}}_a)^\kappa$ contains a converging to 0 super-sequence X topologically generating $(\widehat{\mathbb{Q}}_a)^\kappa$ such that $|X| \leq \sigma$, where $\sigma = \sqrt[\omega]{\kappa}$. Since $(\widehat{\mathbb{Q}}_a)^\kappa$ is dense in $\widehat{\mathbb{Q}}^\kappa$, it follows that X topologically generates $\widehat{\mathbb{Q}}^\kappa$ as well. Therefore, $Y = \pi(X)$ is a super-sequence (by Lemma 4.3(i)) that topologically generates G. Hence $S = Y \setminus \{e_G\}$ is a suitable set for G by Remark 1.11. Clearly, $s(G) \leq |S| \leq |Y| \leq |X| \leq \sigma$. If $\kappa > \mathfrak{c}$, then $s(G) = \sigma$ by Theorem 1.14, which gives $|S| = \sigma$. The remaining case $\kappa \leq \mathfrak{c}$ is settled easily by [15, Lemma 12.32].

(i)→(iii) and (ii)→(iii) Let $S \subseteq G_a$, and let H be the subgroup of G generated by S. If S is a super-sequence that is qc-dense in G (as in case (i)), then H is dense in G by Fact 1.10. If S is a a suitable set for G (as in case (i)), then H is dense in G by the definition of a suitable set. Since $H \subseteq G_a$, it follows that G_a is dense in G as well. Thus G is connected. □

Proof of Theorem 1.3. We have $G = \mathbb{R}^n \times K$, where K is a compact connected group [8, 15]. Since $\mathbb{R}^n$ is arcwise connected, one has $G_a = \mathbb{R}^n \times K_a$. From Theorem 2.1 and Fact 1.5 we conclude that K_a determines K. Hence $G_a = \mathbb{R}^n \times K_a$ determines $G = \mathbb{R}^n \times K$. □

5. The general case: Proofs of Theorems 2.2, 2.5 and 1.15

In the sequel we denote by H' the commutator subgroup of a group H.

Our next lemma shows that qc-density can be essentially studied in the abelian context.

Lemma 5.1. *Let H be a topological group, and let G denote the quotient $H/\overline{H'}$, where $\overline{H'}$ is the closure of H' in H. Let $\pi : H \to G$ denote the canonical map. Then a subset E of H is qc-dense in H if and only if $\pi(E)$ is qc-dense in G.*

Proof. The "only if" part follows from Lemma 3.1. To prove the "if" part, assume that $\pi(E)$ is qc-dense in G, and let $\chi : H \to \mathbb{T}$ be a continuous homomorphism such that $\chi(E) \subseteq \mathbb{T}_+$. Since $\mathbb{T}$ is abelian and Hausdorff, $\overline{H'} \subseteq \ker \chi$, so $\chi = \xi \circ \pi$ for some character $\xi : G \to \mathbb{T}$. Since $\xi(\pi(E)) = \chi(E) \subseteq \mathbb{T}_+$ and $\pi(E)$ is qc-dense in G, we conclude that ξ is trivial, and so χ is trivial too. This proves that E is qc-dense in H. □

Recall that a group L is called *perfect* if $L' = L$.

Corollary 5.2. *Let L be a perfect topological group, G an abelian topological group and $H = G \times L$. Then a subset E of G is qc-dense in G if and only if the subset $E \times \{e_L\}$ of the group H is qc-dense in H.*

Proof. Since $H' = \{0_G\} \times L$, we have $H' = \overline{H'}$ and $G \cong H/H' = H/\overline{H'}$. Now the conclusion of our corollary follows from Lemma 5.1 applied to the projection $\pi : H \to G$. □

Example 5.3. Let $T = \left\{\frac{1}{2n} : n \in \mathbb{N}, n \geq 1\right\} \cup \{0\}$. Assume that L is a finite simple non-abelian group and $G = \mathbb{T} \times L$. Then *$S = \varphi(T) \times \{e_L\}$ is a qc-dense sequence in G converging to e_G*. Indeed, since all simple non-abelian groups are perfect, this follows from Example 3.2 and Corollary 5.2.

Here we recall briefly some basic facts about the structure of compact connected groups that will be used in the last three proofs. According to Levi-Mal′cev structure theorem [15, Theorem 9.24], a compact connected group G satisfies

$$G = c(Z(G))\, G'. \tag{5}$$

A *semisimple* group is a perfect compact connected group [15, Definition 9.5]. Every semisimple group can be presented as a quotient of a product of simple compact Lie groups [15, Theorem 9.19], so it is perfect and arc-wise connected. By Goto's theorem [15, Theorem 9.2], the commutator subgroup G' of a compact connected group G is again compact and connected. Moreover, G' is semisimple [15, Corollary 9.6], hence G is perfect and $G' = G'_a$.

Proof of Theorem 2.2. (i)$\to$(iv) Let $S \subseteq G_a$ be a suitable set for G. Then the subgroup H of G generated by S is dense in G. Since $H \subseteq G_a$, it follows that G_a is dense in G as well. Thus G is connected.

(iv)$\to$(iii) Let us note first, that if $c(Z(G))$ is trivial, then $G = G'$ is semisimple, so by Lemma 5.1 $S = \{e_G\}$ is qc-dense in G. This is why we assume from now on that $c(Z(G))$ is non-trivial. Since this is a connected group, by Theorem 2.1, there exist a super-sequence S convergent to e_G in $c(Z(G))_a$ that qc-dense in $c(Z(G))$ with $|S| \le w(c(Z(G)))$. By Fact 1.10, S topologically generates $c(Z(G))$. Finally, from (5) we deduce that the surjective continuous homomorphism $\psi : G \to G/G'$ sends $c(Z(G))$ onto G/G'. Therefore, $\psi(S)$ is qc-dense in G/G' by Fact 3.1. Now by Lemma 5.1 S is qc-dense in G.

(iii)$\to$(ii) By Theorem 1.13 the subgroup $\{0\} \times L$ has a suitable set $S_0 \subseteq \{0\} \times L$. If $c(Z(G))$ is trivial, this yields $G = G' = G_a$, so by Lemma 5.1 S_0 is qc-dense in G. If S_0 is infinite, we are done. Assume S_0 is finite. There exists a continuous map $f : [0,1] \to G_a$ such that $f(0) = e_G$ and $f(1) \neq e_G$, so we can easily choose an infinite sequence X_0 in $f([0,1])$ converging to 0. By Remark 1.11, $X_0 \setminus \{e_G\}$ is an infinite suitable set contained in $G_a = G$. Now $S = (S_0 \cup X_0) \setminus \{e_G\}$ is as required.

Let us assume now that $c(Z(G))$ is non-trivial. Then by (iii) there exists a super-sequence X in G_a convergent to e_G that is qc-dense in G and that topologically generates $c(Z(G))$. Hence by Remark 1.11, $S_1 = S_0 \cup \{e_G\} \subseteq G'$ is a super-sequence that topologically generates G' and converges to e_G whenever S_0 is infinite. Then $S = X \cup S_1$ is a super-sequence converging to e_G that is qc-dense in G and topologically generates G by (5). So $S \setminus \{e_G\}$ is an infinite suitable set contained in $G_a = G$.

The implication (ii)$\to$(i) is trivial. □

Proof of Theorem 1.15. Let G be a compact connected group. If $w(G) \le \mathfrak{c}$, then the proof of Theorem 1.15 is very easy (see [15, Lemma 12.32]), so we will only deal with the remaining difficult case $w(G) > \mathfrak{c}$. Define $\kappa = w(G)$ and $\sigma = \sqrt[\omega]{\kappa}$. Then $s(G) = \sigma$ by Theorem 1.14(ii). Applying implication (iii)$\to$(ii) of Theorem 2.1, we conclude that $c(Z(G))_a$ contains a suitable set S_1 for $c(Z(G))$ of size $\le \sigma$. From Theorem 1.14(i) it follows that the compact connected group G' contains a suitable set S_2 for G' such that $|S_2| = s(G') \le \sqrt[\omega]{w(G')} \le \sqrt[\omega]{w(G)} = \sigma$. Now (5) yields that $S = S_1 \cup S_2$ is a suitable set for G of size $\le \sigma$. It remains to note that $G' \subseteq G_a$, so $S \subseteq G_a$. □

Lemma 5.4. *Suppose that N is a totally disconnected closed subgroup of a compact connected abelian group K. Then $w(K/N) = w(K)$.*

Proof. Let X be the (discrete) Pontryagin dual of K and $Y = \{\chi \in X : \chi(N) \subseteq \{0\}\}$ the annihilator of N in X. By the standard properties of Pontryagin duality, the Pontryagin dual of K/N is isomorphic to Y and the Pontryagin dual of N is isomorphic to X/Y. Since N is totally disconnected, the quotient group X/Y is torsion [8, Corollary 3.3.9]. Since K is connected, X is torsion-free [8, Corollary 3.3.8]. This yields $|X| = |Y|$. Finally note that $w(K) = |X|$ and $w(K/N) = |Y|$ by [8, Exercise 3.8.23]. □

Proof of Theorem 2.5. First of all, let us show that

$$qcw(G/G') = w(G/G'). \tag{6}$$

Indeed, if $G = G'$, then (6) trivially holds. Otherwise G/G' must be infinite, being a non-trivial continuous image of the compact connected group G. Now Theorem 1.6 applied to the compact abelian group G/G' gives (6).

According to [15, Theorem 9.24], $\Delta = c(Z(G)) \cap G'$ is totally disconnected. Hence $w(c(Z(G)))/\Delta) = w(c(Z(G)))$ by Lemma 5.4. Note that (5) implies the isomorphism $G/G' \cong c(Z(G))/\Delta$, which gives

$$w(G/G') = w(c(Z(G))). \tag{7}$$

From Fact 3.1 and Lemma 5.1 it follows that $qcw(G/G') \leq qcw(G)$. Combining this with (6) and (7), one gets $w(c(Z(G)) \leq qcw(G)$. The converse inequality $qcw(G) \leq w(c(Z(G))$ follows from item (iii) of Theorem 2.2. □

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