Selected topics from the structure theory of topological groups

Dikran Dikranjan and Dmitri Shakhmatov

This article contains open problems and questions covering the following topics: the dimension theory of topological groups, pseudocompact and countably compact group topologies on Abelian groups, with or without nontrivial convergent sequences, categorically compact groups, sequentially complete groups, the Markov–Zariski topology, the Bohr topology, and transversal group topologies. *All topological groups considered in this chapter are assumed to be Hausdorff.*

1. Dimension theory of topological groups

We highlight here our favourite problems from the dimension theory of topological groups.

Problem 1 ([**1**]). *Is* $\operatorname{ind} G = \operatorname{Ind} G = \dim G$ *for a topological group* G *with a countable network?* 884?

The classical result of Pasynkov says that $\operatorname{ind} G = \operatorname{Ind} G = \dim G$ for a (locally) compact group G [**50**].

Question 2 ([**52**]). *Is* $\operatorname{ind} G = \operatorname{Ind} G = \dim G$ *for a* σ*-compact group* G*?* 885?

This is a delicate question since there exists an example of a precompact topological group G such that G is a Lindelöf Σ-space, $\dim G = 1$ but $\operatorname{ind} G = \operatorname{Ind} G = \infty$ [**52**, **53**]. Even the following particular case of Question 2 seems to be open.

Question 3 (M.J. Chasco). *If a topological group* G *is a* k_ω*-space, must* $\operatorname{ind} G = \operatorname{Ind} G = \dim G$*?* 886?

Recall that X is a *k_ω-space* provided that there exists a countable family $\{K_n : n \in \omega\}$ of compact subspaces of X such that a subset U of X is open in X if and only if $U \cap K_n$ is open in K_n for every $n \in \omega$.

Question 4 ([**53**]). *Is* $\operatorname{ind} G = \operatorname{Ind} G$ *for a Lindelöf group* G*?* 887?

The answer to Question 4 is positive if G is a Lindelöf Σ-space (in particular, a σ-compact space), so only the inequality $\operatorname{ind} G \leq \dim G$ must be proved in order to answer Questions 2 or 3 positively.

The first named author was partially supported by the project MIUR 2005 “Anelli commutativi e loro moduli: teoria moltiplicativa degli ideali, metodi omologici e topologici.” The second named author acknowledges partial financial support from the Grant-in-Aid for Scientific Research no. 155400823 by the Japan Society for the Promotion of Science.

888? **Problem 5** (Old problem). *If H is a subgroup of a topological group G, is then $\dim H \le \dim G$?*

The answer is positive if H is $\mathbb{R}$-factorizable [**60**] (in particular, precompact [**54**]).

889? **Question 6** ([**55**]). *Suppose that X is a separable metric space with $\dim X \le n$. Is there a separable metric group G that contains X as a closed subspace such that $\dim G \le 2n+1$?*[1]

Without the requirement that X is closed in G the answer is positive due to the Nöebeling–Pontryagin theorem: X is a subspace of the topological group $\mathbb{R}^{2n+1}$. The separability in the above question is essential: There exists a metric space X of weight ω_1 such that $\dim X = 1$ and X cannot be embedded into any finite-dimensional topological group [**42**].

The next question is the natural group analogue of the classical result about the existence of the universal space of a given weight and covering dimension.

890? **Question 7** ([**55**]). *Let τ be an infinite cardinal and n be a natural number. Is there an (Abelian) topological group $H_{\tau,n}$ with $\dim H_{\tau,n} \le n$ and $w(H_{\tau,n}) \le \tau$ such that every (Abelian) topological group G satisfying $\dim G \le n$ and $w(G) \le \tau$ is topologically and algebraically isomorphic to a subgroup of $H_{\tau,n}$?*

The special case of the above question when $\tau = \omega$ is due to Arhangel′skiĭ [**1**].

Transfinite inductive dimensions have many peculiar properties in topological groups [**56**]. For example, (i) if G is a locally compact group having small transfinite inductive dimension $\mathrm{trind}(G)$, then G must be finite-dimensional, and (ii) if G is a separable metric group having large transfinite inductive dimension $\mathrm{trInd}(G)$, then G must be finite-dimensional as well. It is not clear if (ii) holds for $\mathrm{trind}(G)$ instead of $\mathrm{trInd}(G)$:

891? **Problem 8** ([**56**]). *For which ordinals α does there exist a separable metric group G_α whose small transfinite inductive dimension $\mathrm{trind}(G_\alpha)$ equals α?*

The reader is referred to [**41**, **55**, **56**] for additional open problems in the dimension theory of topological groups.

2. Pseudocompact and countably compact group topologies on Abelian groups

We denote by $\mathcal{C}$ the class of Abelian groups that admit a countably compact group topology, and use $\mathcal{P}$ to denote the class of Abelian groups that admit a pseudocompact group topology.

The next two problems are the most fundamental problems in this area:

[1] We were kindly informed by T. Banakh that Question 6 has been recently answered in the negative in [**2**]: There exists a 1-dimensional separable metric space (namely, the hedgehog with countably many spines) which cannot be embedded into any finite-dimensional topological group as a closed subspace.

Problem 9 ([**19**])**.** *Describe the algebraic structure of members of the class $\mathcal{P}$.* 892?

Problem 10. *Describe the algebraic structure of members of the class $\mathcal{C}$.* 893?

Despite a substantial progress on Problem 9 for particular classes of groups achieved in [**7**, **8**, **16**–**18**], the general case is still very far from the final solution. (We refer the reader to [**4**] for further reading on this topic.)

Let G be an Abelian group. As usual $r(G)$ denotes the *free rank* of G. For every natural number $n \geq 1$ define $G[n] = \{g \in G : ng = 0\}$ and $nG = \{ng : g \in G\}$. Every group G from the class $\mathcal{C}$ satisfies the following two conditions [**16**, **18**, **26**]:

PS: Either $r(G) \geq \mathfrak{c}$ or $G = G[n]$ for some $n \in \omega \setminus \{0\}$.

CC: For every pair of integers $n \geq 1$ and $m \geq 1$ the group $mG[n]$ is either finite or has size at least $\mathfrak{c}$.

It is totally unclear if these are the only necessary conditions on a group from the class $\mathcal{C}$:

Question 11. *Is it true that an Abelian group G belongs to $\mathcal{C}$ if and only if G satisfies both* **PS** *and* **CC***?* 894?

Question 12 ([**22**])**.** *Is it true in* ZFC *that an Abelian group G of size at most $2^{\mathfrak{c}}$ belongs to $\mathcal{C}$ if and only if G satisfies both* **PS** *and* **CC***?* 895?

Question 12 has a positive *consistent* answer [**22**].

Assuming MA, there exist countably compact Abelian groups G, H such that $G \times H$ is not countably compact [**31**]. Therefore, our next question could be viewed as a *weaker form* of productivity of countable compactness in topological groups that still has a chance for a positive answer in ZFC.

Question 13 ([**19**])**.** *If G and H belong to $\mathcal{C}$, must then their product $G \times H$ also belong to $\mathcal{C}$?* 896?

In fact, one can consider a much bolder hypothesis:

Question 14 ([**19**])**.** *Is $\mathcal{C}$ closed under* arbitrary *products? That is, if G_i belongs to $\mathcal{C}$ for each $i \in I$, does then $\prod_{i\in I} G_i$ belong to $\mathcal{C}$?* 897?

The next question provides a slightly less bold conjecture:

Question 15 ([**19**])**.** (i) *Is there a cardinal τ having the following property: A product $\prod_{i\in I} G_i$ belongs to $\mathcal{C}$ provided that $\prod_{j\in J} G_j$ belongs to $\mathcal{C}$ whenever $J \subseteq I$ and $|J| \leq \tau$?* 898–899?

(ii) *Does the statement in item* (i) *hold true when $\tau = \mathfrak{c}$ or $\tau = 2^{\mathfrak{c}}$?*

Of course Question 14 simply asks if the statement in item (i) of Question 15 holds true when $\tau = 1$. It might be worth noting that Question 15 is motivated by a theorem of Ginsburg and Saks [**35**]: A product $\prod_{i\in I} X_i$ of topological spaces X_i is countably compact provided that $\prod_{j\in J} G_j$ is countably compact whenever $J \subseteq I$ and $|J| \leq 2^{\mathfrak{c}}$.

A partial positive answer to Question 14 has been given in [**20**]: It is consistent with ZFC that, for every family $\{G_i : i \in I\}$ of groups with $2^{|I|} \leq 2^{\mathfrak{c}}$ such that

G_i belongs to $\mathcal{C}$ and $|G_i| \le 2^{\mathfrak{c}}$ for each $i \in I$, the product $\prod_{i\in I} G_i$ also belongs to $\mathcal{C}$. A similar result for smaller products and smaller groups has been proved in [**26**, Theorem 5.6] under the assumption of MA. In particular, if the groups G and H in Question 13 are additionally assumed to be of size at most $2^{\mathfrak{c}}$, then the positive answer to this restricted version of Question 13 is consistent with ZFC [**20**].

Recall that an Abelian group G is *algebraically compact* provided that one can find an Abelian group H such that $G \times H$ admits a compact group topology. Algebraically compact groups form a relatively narrow subclass of Abelian groups (for example, the group $\mathbb{Z}$ of integers is not algebraically compact). On the other hand, every Abelian group G is *algebraically pseudocompact*; that is, one can find an Abelian group H such that $G \times H \in \mathcal{P}$ [**18**, Theorem 8.15]. It is unclear if this result can be strengthened to show that every Abelian group is *algebraically countably compact*:

900? **Question 16** ([**22**]). *Given an Abelian group G, can one always find an Abelian group H such that $G \times H \in \mathcal{C}$?*

Recall that an Abelian group G is *divisible* provided that for every $g \in G$ and each positive integer n one can find $h \in G$ such that $nh = g$. An Abelian group is *reduced* if it does not have non-zero divisible subgroups. Every Abelian group G admits a unique representation $G = D(G) \times R(G)$ into the maximal divisible subgroup $D(G)$ of G (the so-called *divisible part of* G) and the reduced subgroup $R(G) \cong G/D(G)$ of G (the so-called *reduced part of* G). It is well-known that an Abelian group G admits a compact group topology if and only if both its divisible part $D(G)$ and its reduced part $R(G)$ admit a compact group topology. However, there exist groups G and H that belong to $\mathcal{P}$ but neither $D(G)$ nor $R(H)$ belong to $\mathcal{P}$ [**18**, Theorem 8.1(ii)]. This was *strengthened* in [**22**, **26**] as follows: It is consistent with ZFC that there exist groups G' and H' from the class $\mathcal{C}$ such that neither $D(G')$ nor $R(H')$ belong to $\mathcal{P}$. These results leave open the following:

901–902? **Problem 17** ([**19**]). *In* ZFC*, give an example of groups G and H from the class $\mathcal{C}$ such that:*

(i) *$D(G)$ does not belong to $\mathcal{C}$ (or even $\mathcal{P}$),*
(ii) *$R(H)$ does not belong to $\mathcal{C}$ (or even $\mathcal{P}$).*

Even the following question is also open:

903–904? **Question 18** ([**19**]). *Let G be a group in $\mathcal{C}$.*

(i) *Is it true that either $D(G)$ or $R(G)$ belongs to $\mathcal{C}$?*
(ii) *Must either $D(G)$ or $R(G)$ belong to $\mathcal{P}$?*

We note that item (ii) of the last question is a strengthening of Question 9.8 from [**18**]. Even consistent results related to the last question are currently unavailable.

An Abelian group G is *torsion* provided that $G = \bigcup\{G[n] : n \in \omega, n \ge 1\}$, and is *torsion-free* provided that $\bigcup\{G[n] : n \in \omega, n \ge 1\} = \{0\}$.

Question 19 ([**22**])**.** *Is there a torsion Abelian group that is in $\mathcal{P}$ but not in $\mathcal{C}$?* 905?

Question 20 ([**22**])**.** *Is there a torsion-free Abelian group that is in $\mathcal{P}$ but not in $\mathcal{C}$?* 906?

It is consistent with ZFC that a group Questions 19 and 20 must have size strictly bigger than $2^{\mathfrak{c}}$ [**22**].

Problem 21 ([**22**])**.** (i) *Describe in* ZFC *the algebraic structure of separable countably compact Abelian groups.* 907–908?

(ii) *Is it true in* ZFC *that an Abelian group G admits a separable countably compact group topology if and only if $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both* **PS** *and* **CC***?*

A consistent positive solution to Problem 21(ii) is given in [**22**].

3. Properties determined by convergent sequences

It is well-known that infinite compact groups have (lots of) nontrivial convergent sequences. There exists an example (in ZFC) of a pseudocompact Abelian group without nontrivial convergent sequences [**58**]. While there are plenty of consistent examples of countably compact groups without nontrivial convergent sequences [**10**, **22**, **26**, **31**, **39**, **46**, **59**, **63**], the following remains a major open problem in this area:

Problem 22. *Does there exist, in* ZFC*, a countably compact group without nontrivial convergent sequences?* 909?

Recall that a (Hausdorff) topological group G is *minimal* if G does not admit a strictly weaker (Hausdorff) group topology. Even though a countably compact, minimal Abelian group need not be compact, it can be shown that it must contain a nontrivial convergent sequence. More generally, one can show that an infinite, countably compact, minimal nilpotent group has a nontrivial convergent sequence. Whether "nilpotent" can be dropped remains unclear.

Problem 23. *Must an infinite, countably compact, minimal group contain a nontrivial convergent sequence?* 910?

The next question may be considered as a countably compact (or pseudocompact) *heir* of the fact that compact groups have nontrivial convergent sequences that still has a chance of a positive answer in ZFC.

Question 24 ([**22**])**.** *Let G be an infinite group admitting a countably compact (or a pseudocompact) group topology. Does G have a countably compact (respectively, pseudocompact) group topology that contains a nontrivial convergent sequence?* 911?

The next question goes in the opposite direction:

Question 25 ([**22**])**.** (i) *Does every group G admitting a pseudocompact group topology have also a pseudocompact group topology without nontrivial convergent sequences (without infinite compact subsets)?* 912–913?

(ii) *Does every group G admitting a countably compact group topology have also a countably compact group topology without nontrivial convergent sequences (without infinite compact subsets)?*

Question 25(ii) has a consistent positive answer in the special case when $|G| \leq 2^{\mathfrak{c}}$ [**22**]. The part "without nontrivial convergent sequences" of item (ii) of our next question has appeared in [**10**].

914–916? **Question 26.** (i) *When does a compact Abelian group G admit a proper dense subgroup H without nontrivial convergent sequences? without infinite compact subsets?*

(ii) *When does a compact Abelian group G admit a proper dense pseudocompact subgroup H without nontrivial convergent sequences? without infinite compact subsets?*

(iii) *When does a compact Abelian group G admit a proper dense countably compact subgroup H without nontrivial convergent sequences? without infinite compact subsets?*

In GCH, a precompact group H such that $w(H) < w(H)^{\omega}$ has a nontrivial convergent sequence [**47**]. Thus $w(G) = w(G)^{\omega}$ is a necessary condition for the group G to have a subgroup H as in Question 26. This condition alone is not sufficient: If $K = \prod_{n\in\omega} \mathbb{Z}_{2^n}$ and τ is an infinite cardinal, then every dense subgroup H of $G = K^{\tau}$ has a nontrivial convergent sequence [**10**] (here $\mathbb{Z}_m$ denotes the cyclic group $\mathbb{Z}/m\mathbb{Z}$). Many partial results towards solution of Question 26 are given in [**10**, **33**].

917–918? **Question 27.** (i) *If a compact Abelian group has a proper dense pseudocompact subgroup without nontrivial convergent sequences, does it also have a proper dense pseudocompact subgroup without infinite compact subsets?*

(ii) *If a compact Abelian group has a proper dense countably compact subgroup without nontrivial convergent sequences, does it also have a proper dense countably compact subgroup without infinite compact subsets?*

Now we relax item (ii) to get the following:

919? **Question 28.** *Is the existence of a countably compact Abelian group without nontrivial convergent sequences equivalent to the existence of a countably compact Abelian group without infinite compact subsets?*

In connection with the last four questions we should note that, under MA, an infinite compact space of size at most $\mathfrak{c}$ contains a nontrivial convergent sequence.

A topological group G is called *sequentially complete* [**24**, **25**] if G is sequentially closed in every (Hausdorff) group that contains G as a topological subgroup. Obviously, every topological group without nontrivial convergent sequences is sequentially complete. Moreover, sequential completeness is preserved under taking arbitrary direct products and closed subgroups [**24**].

Denote by $\mathcal{S}$ the class of closed subgroups of the products of countably compact Abelian groups. Since countably compact groups are sequentially complete and precompact, every group from the class $\mathcal{S}$ is sequentially complete and precompact.

Question 29 ([**25**]). (i) *Does every precompact sequentially complete Abelian group G belong to $\mathcal{S}$?* 920–921?

(ii) *What is the answer to* (i) *if one additionally assumes that $|G| \leq \mathfrak{c}$?*

Every precompact Abelian group is both a quotient group and a continuous isomorphic image of some sequentially complete precompact Abelian group [**25**, Theorem B]. This motivates the following:

Question 30 ([**25**]). *Is every precompact Abelian group G:* 922–924?

(i) *a quotient of a group from $\mathcal{S}$?*
(ii) *a continuous homomoprhic image of group from $\mathcal{S}$?*
(iii) *a continuous isomorphic image of group from $\mathcal{S}$?*

Item (iii) of Question 30 has a positive answer when $|G| \leq \mathfrak{c}$ [**25**, Theorem A], and more generally, if $|G|$ is a non-measurable cardinal [**61**].

4. Categorically compact groups

A topological group G is *categorically compact* (briefly, *c-compact*) if for each topological group H the projection $G \times H \to H$ sends closed subgroups of $G \times H$ to closed subgroups of H [**29**]. Obviously, compact groups are c-compact. To establish the converse is the main open problem in this area:

Problem 31. (i) *Are c-compact groups compact?* 925–926?

(ii) *Are nondiscrete c-compact groups compact?*

Item (i) has appeared in [**29**]. Two related weaker versions are also open:

Question 32. *Is every (nondiscrete) c-compact group minimal?* 927?

Question 33. *Does every nondiscrete c-compact group have a nontrivial convergent sequence?* 928?

A positive answer to Problem 31 in the Abelian case makes recourse to the deep theorem of precompactness of Prodanov and Stoyanov [**15**]. Similar to (usual) compactness, taking products, closed subgroups and continuous homomorphic images preserves c-compactness [**29**] (a proof of the productivity of c-compactness was obtained independently also in [**3**] in a much more general setting). Therefore, a positive answer to Question 32 would imply that every closed subgroup H of a c-compact group is *totally minimal*, i.e., all quotient groups of H are minimal. At present we only know that separable c-compact groups are totally minimal (and complete) [**29**].

Lukacs [**45**] resolved Problem 31 positively for maximally almost periodic groups. Moreover, he showed that it suffices to solve this problem only for second countable groups (analogously, the case of locally compact SIN-groups, is reduced to that of countable discrete groups [**45**]). (Recall that a *SIN-group* is a topological group for which the left and right uniformities coincide.) According to [**45**], in Question 33 it suffices to consider only the nondiscrete c-compact groups that have no nontrivial continuous homomorphisms into compact groups.

Connected locally compact c-compact groups are compact [**29**]. Hence the connected locally compact group $SL_2(\mathbb{R})$ is not categorically compact, although it is separable and totally minimal [**15**]. Nothing is known about c-compactness of disconnected locally compact groups. In fact, even the discrete case is wide open:

929? **Question 34** ([**29**]). *Is every discrete c-compact group finite (finitely generated, of finite exponent, countable)?*

One can prove that a countable discrete group G is c-compact if and only if every subgroup of G is totally minimal [**29**]. Therefore, the negative answer to this question is equivalent to the existence of an infinite group G such that no subgroup or quotient group of G admits a nondiscrete Hausdorff group topology (this is a stronger version of the famous Markov problem on the existence of a countably infinite group without nondiscrete Hausdorff group topologies).

A group G is *h-complete* if all continuous homomorphic images of G are complete, and G is *hereditarily h-complete* if every closed subgroup of G is h-complete. c-compact groups are hereditarily h-complete, and the inverse implication holds for SIN-groups (in particular, Abelian groups) [**29**].

Both c-compactness and h-completeness are stable under products, and h-completeness also has the the so-called "three space property": If K is a closed normal subgroup of a topological group G such that both K and the quotient group G/K are h-complete, then G is h-complete. This leaves open:

930? **Question 35** ([**29**, Question 4.3]). *If K is a closed normal subgroup of a topological group G such that both K and the quotient group G/K are c-compact, must G be c-compact as well?*

Nilpotent (in particular, Abelian) h-complete groups are compact, while solvable c-compact groups are compact [**29**]. This motivates the following:

931? **Question 36** ([**29**, Question 3.13]). *Are solvable h-complete groups c-compact?*

5. The Markov–Zariski topology

According to Markov [**48**], a subset S of a group G is called:

(a) *elementary algebraic* if there exist an integer $n > 0$, $a_1, \dots, a_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that
$$S = \{x \in G : x^{\varepsilon_1} a_1 x^{\varepsilon_2} a_2 \dots a_{n-1} x^{\varepsilon_n} = a_n\},$$
(b) *algebraic* if S is an intersection of finite unions of elementary algebraic subsets,
(c) *unconditionally closed* if S is closed in *every* Hausdorff group topology of G,
(d) *potentially dense* if G admits a Hausdorff group topology $\mathcal{T}$ on G such that S is dense in $(G, \mathcal{T})$.

The family of algebraic subsets of a group G coincides with the family of closed subsets of a T_1 topology $\mathfrak{Z}_G$ on G, called the *Zariski topology*. The family of unconditionally closed subsets of G coincides with the family of closed subsets of a

T_1 topology $\mathfrak{M}_G$ on G, namely the infimum (taken in the lattice of all topologies on G) of all Hausdorff group topologies on G. We call $\mathfrak{M}_G$ the *Markov topology* of G. Analogously, let $\mathfrak{P}_G$ be the infimum of all precompact Hausdorff group topologies on G (if G admits no such topologies let $\mathfrak{P}_G$ denote the discrete topology of G). It seems natural to call $\mathfrak{P}_G$ the *precompact Markov topology* of G. Note that $(G, \mathfrak{Z}_G)$, $(G, \mathfrak{M}_G)$ and $(G, \mathfrak{P}_G)$ are quasi-topological groups, i.e., the inversion and shifts are continuous.

Clearly, $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$. If G is Abelian, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ [**21**]. Markov has attributed the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ in the Abelian case to Perel′man but the proof has never appeared in print. Another proof was recently announced by Sipacheva [**41**]. In the particular case when G is almost torsion-free[2] the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ was earlier proved in [**62**].

Problem 37 ([**48**])**.** *Does $\mathfrak{Z}_G = \mathfrak{M}_G$ hold true for an arbitrary group G?* 932?

The answer is positive when G is countable [**48**]. A consistent counterexample to this question was announced quite recently by Sipacheva [**57**] (see also [**41**]).

Let $\mathcal{M}$ denote the class of groups G with $\mathfrak{Z}_G = \mathfrak{M}_G$.

Question 38. *For which infinite cardinals τ does the permutation group $S(\tau)$ of a set of size τ belong to $\mathcal{M}$?* 933?

The answer is positive for $\tau = \omega$ [**23**].

Question 39. *For which uncountable cardinals τ does the free group of rank τ belong to $\mathcal{M}$?* 934?

Question 40. (i) *Is $\mathcal{M}$ closed under taking subgroups? In particular, do all subgroups of $S(\omega)$ belong to $\mathcal{M}$?* 935–936?

(ii) *Is $\mathcal{M}$ closed under taking (finite) direct products?*

Question 41. (i) *What is the minimal cardinality of a group G such that $\mathfrak{Z}_G \neq \mathfrak{M}_G$?* 937–939?

(ii) *Is $\mathfrak{c}$ such a cardinality in* ZFC*?*

(iii) *Is ω_1 such a cardinality in* ZFC*?*

Let G be a group and $\mathcal{T}$ be any Hausdorff group topology on G. Then $\mathfrak{M}_G \subseteq \mathcal{T}$, and therefore the $\mathcal{T}$-closure of a subset of G must be contained in its $\mathfrak{M}_G$-closure. In other words, the $\mathfrak{M}_G$-closure of a given set $S \subseteq G$ is the *biggest* subset of G that one could possibly hope to attain by taking the closure of S in *any* Hausdorff group topology on G. This naturally leads to a question whether the $\mathfrak{M}_G$-closure of S can actually be *realized* by taking the closure of S in *some* Hausdorff group topology on G.

Question 42. *Let G be a group of size at most $2^{\mathfrak{c}}$ and $\mathcal{E}$ a countable family of subsets of G. Can one find a Hausdorff group topology $\mathcal{T}_{\mathcal{E}}$ on G such that the $\mathcal{T}_{\mathcal{E}}$-closure of every $E \in \mathcal{E}$ coincides with its $\mathfrak{M}_G$-closure?* 940?

[2]An Abelian group G is *almost torsion-free* if $G[n]$ is finite for every $n > 1$.

For an Abelian group G the answer is positive, and in fact the topology $\mathcal{T}_{\mathcal{E}}$ in this case can be chosen to be precompact [**21**].

The counterpart of Question 42 for $\mathfrak{Z}_G$ instead of $\mathfrak{M}_G$ has a negative (consistent) answer. Indeed, let G be an infinite group such that $\mathfrak{Z}_G \neq \mathfrak{M}_G$ and $\mathfrak{M}_G$ is discrete. (The recent example of Sipacheva [**57**] would do.) Let e be the unit element of G. Then $G \setminus \{e\}$ cannot be $\mathfrak{Z}_G$-closed. Indeed, if it were, then $\{e\}$ would be $\mathfrak{Z}_G$-open, and so by homogeneity of $\mathfrak{Z}_G$, the topology $\mathfrak{Z}_G$ would be discrete, implying $\mathfrak{Z}_G = \mathfrak{M}_G$, a contradiction. So the $\mathfrak{Z}_G$-closure of $G \setminus \{e\}$ must coincide with G. On the other hand, the only Hausdorff group topology $\mathcal{T}$ on G is the discrete topology, and so the $\mathcal{T}$-closure of $G \setminus \{e\}$ is $G \setminus \{e\}$.

Let us consider now the counterpart of Question 42 for $\mathfrak{P}_G$ instead of $\mathfrak{M}_G$.

941? **Question 43.** *Let G be a group of size at most $2^{\mathfrak{c}}$ having at least one precompact Hausdorff group topology, and let $\mathcal{E}$ be a countable family of subsets of G. Can one find a precompact Hausdorff group topology $\mathcal{T}_{\mathcal{E}}$ on G such that the $\mathcal{T}_{\mathcal{E}}$-closure of every $E \in \mathcal{E}$ coincides with its $\mathfrak{P}_G$-closure?*

Again, for an Abelian group G the answer is positive [**21**].

Thereafter, we consider only Abelian groups and refer to the three equivalent topologies $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ as the *Markov–Zariski topology*, denoting it by $\mathfrak{T}_G$. For an infinite Abelian group G, $\mathfrak{T}_G$ is neither Hausdorff, nor a group topology on G, but still has various nice properties, e.g., the space $(G, \mathfrak{T}_G)$ is hereditarily compact, hereditarily separable and Fréchet–Urysohn, moreover it has only finitely many connected components, and each component is an irreducible space [**21**].

942? **Problem 44.** *Let G be an Abelian group with $|G| \leq 2^{\mathfrak{c}}$ and $\mathcal{E}$ a family of subsets of G with $|\mathcal{E}| < 2^{|G|}$. Does there exist a precompact Hausdorff group topology $\mathcal{T}_{\mathcal{E}}$ on G such that the $\mathcal{T}_{\mathcal{E}}$-closure of each $E \in \mathcal{E}$ coincides with its $\mathfrak{T}_G$-closure?*

As was mentioned before, the answer is positive for countable families $\mathcal{E}$ [**21**]. Moreover, it was shown that if $|G| \leq \mathfrak{c}$, one can choose the approximating topology $\mathcal{T}_{\mathcal{E}}$ to be even metric.

The inequality $|\mathcal{E}| < 2^{|G|}$ in the above problem is essential. Indeed, let G be an infinite Abelian group. If one takes as $\mathcal{E}$ the family of *all* subsets of G, then the existence of a Hausdorff group topology $\mathcal{T}_{\mathcal{E}}$ on G such that the $\mathcal{T}_{\mathcal{E}}$-closure of each $E \in \mathcal{E}$ coincides with its $\mathfrak{T}_G$-closure would obviously imply that $\mathcal{T}_{\mathcal{E}} = \mathfrak{T}_G$. Thus $\mathfrak{T}_G$ would become Hausdorff, a contradiction.

The restriction on the cardinality of the group G in Questions 42, 43 and Problem 44 is obviously necessary since the closure of a countable set in a Hausdorff topology cannot exceed $2^{\mathfrak{c}}$.

The problem of characterization of the potentially dense subsets S of a group G goes back to Markov [**48**] who proved that every infinite subset of $\mathbb{Z}$ is potentially dense. This was extended in [**62**] to Abelian groups G of size $\leq \mathfrak{c}$ that are either of prime exponent or almost torsion-free. Tkachenko and Yaschenko asked in [**62**] whether the restriction $|G| \leq \mathfrak{c}$ can be relaxed to $|G| \leq 2^{\mathfrak{c}}$. To clarify better the problem, let us drop all additional restrictions on the Abelian group G. Obviously, if S is potentially dense in G, then $|G| \leq 2^{2^{|S|}}$ and S must be $\mathfrak{T}_G$-dense in G. It is not clear if these two conditions are not only necessary but also sufficient for potential density.

Question 45. *Let G be an Abelian group and let S be an infinite subset of G such that $|G| \leq 2^{2^{|S|}}$ and S is $\mathfrak{T}_G$-dense in G.* 943–944?

(i) *Is S potentially dense in G?*
(ii) *Does there exist a Hausdorff precompact group topology $\mathcal{T}$ such that S is $\mathcal{T}$-dense in G?*

A positive answer to both items of this question in the case of a countable set S has been given in [**21**] thereby providing a positive answer to the above mentioned question from [**62**].

6. Bohr topologies of Abelian groups

Let G be an Abelian group. Following E. van Douwen [**32**], we denote by $G^{\#}$ the group G equipped with the Bohr topology, i.e., the initial topology with respect to the family of all homomorphisms of G into the circle group $\mathbb{T}$. It is a well known fact, due to Glicksberg (see also [**34**] in this volume), that $G^{\#}$ has no infinite compact subsets (in particular, no nontrivial convergent sequences). Therefore, $G^{\#}$ is always sequentially complete. For future reference, we mention two fundamental properties of the Bohr topology for arbitrary Abelian groups G, H:

(i) the Bohr topology of $G \times H$ coincides with the product topology of $G^{\#} \times H^{\#}$;
(ii) if H is a subgroup of G, then H is closed in $G^{\#}$ and its topology as a topological subgroup of $G^{\#}$ coincides with that of $H^{\#}$.

E. van Douwen [**49**] posed the following challenging problem (see also [**34**]): If G and H are Abelian groups of the same size, must $G^{\#}$ and $H^{\#}$ be homeomorphic? A negative solution was obtained in [**43**] and independently, in [**30**]: $(\mathbb{V}_p^{\omega})^{\#}$ and $(\mathbb{V}_q^{\omega})^{\#}$ are not homeomorphic for different primes p and q. (For every positive integer m and a cardinal κ, $\mathbb{V}_m^{\kappa}$ denotes the direct sum of κ many copies of the group $\mathbb{Z}_m$.) Motivated by this, let us call a pair G, H of infinite Abelian groups:

(1) *Bohr-homeomorphic* if $G^{\#}$ and $H^{\#}$ are homeomorphic,
(2) *weakly Bohr-homeomorphic* if $G^{\#}$ can be homeomorphically embedded into $H^{\#}$ and $H^{\#}$ can also be homeomorphically embedded into $G^{\#}$.

Obviously, Bohr-homeomorphic groups are weakly Bohr-homeomorphic, and the status of the converse implication is totally unclear (see Question 49(ii)). As we shall see in the sequel, weak Bohr-homeomorphism provides a more flexible tool for studying the Bohr topology than the more *rigid* notion of Bohr-homeomorphism, e.g, $(\mathbb{V}_p^{\omega})^{\#}$ and $(\mathbb{V}_q^{\omega})^{\#}$ are not even weakly Bohr-homeomorphic for different primes p and q.

If $G^{\#}$ homeomorphically embeds into $H^{\#}$ and H is a bounded torsion group, then G must also be a bounded torsion group [**37**]. In particular, boundedness is invariant under weak Bohr-homeomorphisms, i.e., if G is a bounded Abelian group and the pair G, H are weakly Bohr-homeomorphic, than H must be bounded. Therefore, when studying weak Bohr-homeomorphisms (and thus Bohr-homeomorphisms), without any loss of generality whatsoever, one can consider completely separately bounded torsion Abelian groups and non-bounded Abelian groups.

We start first with the class of bounded torsion Abelian groups. According to Prüfer's theorem, every infinite bounded group has the form $\prod_{i=1}^{n} \mathbb{V}_{m_i}^{\kappa_i}$ for certain integers $m_i > 0$ and cardinals κ_i. For this reason, and in view of items (i) and (ii), the study of the Bohr topology of the bounded Abelian groups can be focused on the groups $\mathbb{V}_m^{\kappa}$.

For bounded Abelian groups G, H the following two algebraic conditions play a prominent role.

(3) $|mG| = |mH|$ whenever $m \in \mathbb{N}$ and $|mG| \cdot |mH| \geq \omega$.

(4) $\mathrm{eo}(G) = \mathrm{eo}(H)$ and $r_p(G) = r_p(H)$ for all p with $r_p(G) + r_p(H) \geq \omega$, where $\mathrm{eo}(G)$ is the *essential order* of G [**9**, **37**], i.e., *the smallest positive integer* m with mG finite.

Since a pair G, H satisfies (3) iff each one of these groups has a finite-index subgroup that is isomorphic to a subgroup of the other [**9**], we call such pairs of bounded Abelian groups G and H *weakly isomorphic* [**9**]. By (ii), weakly isomorphic bounded Abelian groups are weakly Bohr-homeomorphic. According to [**9**], weakly Bohr-homeomorphic bounded Abelian groups satisfy (4), i.e.,

weakly isomorphic $\Rightarrow$ weakly Bohr-homeomorphic $\Rightarrow$ (4).

Let us discuss the opposite implications. For countable Abelian groups G, H the second part of (4) becomes vacuous, while $eo(G) = eo(H)$ yields that G, H are weakly isomorphic. Analogously, one can see that (4) for groups of square-free essential order implies weak isomorphism and Bohr-homeomorphism. Hence all four properties (1)–(4) coincide for bounded Abelian groups that have square-free essential order [**9**, **37**]. Therefore, the invariant $\mathrm{eo}(G)$ alone allows for a complete classification (up to Bohr-homeomorphism) of all bounded Abelian groups of this class.

The situation changes completely even for the simplest *uncountable* bounded Abelian groups of essential order 4. Indeed, $G = \mathbb{V}_4^{\omega_1}$ and $H = \mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^{\omega}$ are not weakly isomorphic, because $\omega_1 = |2G| > |2H| = \omega$. However, we do not know whether these groups are weakly Bohr-homeomorphic:

945? **Question 46.** *Can* $(\mathbb{V}_4^{\omega_1})^{\#}$ *be homeomorphically embedded into* $(\mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^{\omega})^{\#}$*?*

Here is the question in the most general form:

946? **Question 47.** *Given a cardinal* $\kappa \geq \omega$ *and an integer* $s > 1$*, are* $\mathbb{V}_{p^s}^{\kappa}$ *and* $\mathbb{V}_p^{\kappa} \times \mathbb{V}_{p^s}^{\omega}$ *weakly Bohr-homeomorphic? Can this depend on* p*?*

If the answer to Question 47 is positive for all p, then bounded Abelian groups G and H would be weakly Bohr-homeomorphic if and only if (4) holds.

The next question is an equivalent form of the strongest negative answer to Question 47.

Question 48. *Assume that p is a prime number, $k > 1$ is an integer, κ and λ are infinite cardinals such that $(\mathbb{V}_{p^k}^{\kappa})^{\#}$ can be homeomorphically embedded into $(\mathbb{V}_{p^{k-1}}^{\kappa} \times \mathbb{V}_{p^k}^{\lambda})^{\#}$. Must then inequality $\lambda \geq \kappa$ hold?* 947?

Note that a positive answer to Question 48 answers negatively Questions 46 and 47.

The countable groups $\mathbb{V}_4^{\omega}$ and $\mathbb{V}_2^{\omega} \times \mathbb{V}_4^{\omega}$ are obviously weakly isomorphic, hence weakly Bohr-homeomorphic (see the discussion above).

Question 49. (i) ([**43**]) *Are $\mathbb{V}_4^{\omega}$ and $\mathbb{V}_2^{\omega} \times \mathbb{V}_4^{\omega}$ Bohr-homeomorphic?* 948–949?
(ii) *Are weakly Bohr-homeomorphic bounded groups* always *Bohr-homeomorphic?*

Question 50. *Suppose that G and H are bounded Abelian groups such that $G^{\#}$ homeomorphically embeds into $H^{\#}$. Does there exist a subgroup G' of G of finite index that algebraically embeds into H?* 950?

Note that a positive answer to this question would imply, in particular, that weak Bohr-homeomorphism coincides with weak isomorphism. Hence a positive answer to this question would imply a positive answer to Question 48.

Now we leave the *bounded world* and turn to the class of non-bounded groups. According to Hart and Kunen [**40**], two Abelian groups G and H are *almost isomorphic* if G and H have isomorphic finite index subgroups. This definition is motivated by the fact that almost isomorphic Abelian groups are always Bohr-homeomorphic [**40**]. The converse implication fails. Indeed, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ are Bohr-homeomorphic [**6**], and yet these groups are not almost isomorphic. It is nevertheless unclear if the reverse implication holds for bounded groups.

Question 51 ([**43**])**.** *Are Bohr-homeomorphic bounded Abelian groups almost isomorphic?* 951?

The question on whether the pairs $\mathbb{Z}, \mathbb{Z}^2$ and $\mathbb{Z}, \mathbb{Q}$ are Bohr-homeomorphic is raised in [**5**, **34**]. Let us consider here the version for weak Bohr-homeomorphisms:

Question 52. (i) *Are $\mathbb{Z}$ and $\mathbb{Q}$ weakly Bohr-homeomorphic?* 952–953?
(ii) *Are $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ (weakly) Bohr-homeomorphic?*

A positive answer to item (i) of Question 52 would yield that all torsion-free Abelian groups of a fixed finite free rank are weakly Bohr-homeomorphic. If both items have a positive answer, then the weak Bohr-homeomorphism class of $\mathbb{Z}^{\#}$ would comprise the class of all Abelian groups G of finite rank[3] such that either G is non-torsion or G contains a copy of the group $\mathbb{Q}/\mathbb{Z}$. (In particular, all finite powers of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ along with their finite products would become weakly Bohr-homeomorphic.)

Many nice properties of $\mathbb{Z}^{\#}$ can be found in [**44**]. For a fast growing sequence a_n in $\mathbb{Z}^{\#}$ the range is a closed discrete set of $\mathbb{Z}^{\#}$ (see [**34**] for further properties of the lacunary sets in $\mathbb{Z}^{\#}$), whereas for a polynomial function $n \mapsto a_n = P(n)$ the range has no isolated points [**44**, Theorem 5.4]. Moreover, the range $P(\mathbb{Z})$ is closed

[3]I.e., there exists $n \in \omega$ such that $r_0(G) \leq n$ and $|G[p]| \leq p^n$ for every prime p.

when $P(x) = x^k$ is a monomial. For quadratic polynomials $P(x) = ax^2 + bx + c$ ($a, b, c \in \mathbb{Z}$, $a \neq 0$) the situation is already more complicated: the range $P(\mathbb{Z})$ is closed iff there is at most one prime that divides a, but does not divide b [**44**, Theorem 5.6]. This leaves open the general question.

954? **Problem 53.** *Characterize the polynomials $P(x) \in \mathbb{Z}[x]$ such that $P(\mathbb{Z})$ is closed in $\mathbb{Z}^{\#}$.*

Answering a question of van Douwen, Gladdines [**38**] found a closed countable *subset* of $(\mathbb{V}_2^{\omega})^{\#}$ that is not a retract of $(\mathbb{V}_2^{\omega})^{\#}$, while Givens [**36**] proved that every infinite $G^{\#}$ contains a closed countable subset that is not a retract of $G^{\#}$. However, the question remains open in the case of *subgroups*:

955? **Question 54** (Question 81, [**49**]). *If H is a countable subgroup of an Abelian group G, must $H^{\#}$ be a retract of $G^{\#}$?*

An affirmative answer to this question of E. van Douwen was obtained in [**6**] in the case when H is finitely generated (see also [**13**] for other partial results and open problems). The general case is still open.

We refer the reader to [**12**, **14**] for further information about Bohr topology.

7. Miscellanea

Two nondiscrete topologies τ_1 and τ_2 on a set X are called *transversal* if $\tau_1 \cup \tau_2$ generates the discrete topology on X. A precompact group topology on a group does not admit a transversal group topology, and under certain natural conditions the converse is also true [**27**].

956? **Question 55** ([**28**]). *Characterize locally compact groups that admit a transversal group topology.*

This question is resolved for locally compact Abelian groups [**27**] and for connected locally compact groups [**28**].

There exists a locally Abelian group G and a compact normal subgroup K of G such that G does not admit a transversal group topology while G/K does have a transversal group topology [**27**, Example 5.4]. The inverse implication remains unclear:

957? **Question 56** ([**28**]). *If G is a topological group that admits a transversal group topology and K is a compact normal subgroup of G, does also G/K admit a transversal group topology?*

The answer is positive when $G = K \times H$ for some subgroup H of G [**27**], or when G is a locally compact Abelian group (argue as in the proof of the implication (d) $\Rightarrow$ (c) of [**27**, Corollary 6.7]).

958–959? **Question 57** ([**28**]). (i) *Is it true that no minimal group topology admits a transversal group topology?*

(ii) *Does the topology of the unitary group of an infinite-dimensional Hilbert space admit a transversal group topology?*

The answer to item (i) is positive in the Abelian case.

The quasi-components (respectively, the connected components) of the Abelian pseudocompact groups are precisely all (connected) precompact groups [**11**]. The non-Abelian case remains unclear:

Problem 58 ([**11**]). *Describe the connected components and the quasi-components of pseudocompact groups.* 960?

Given a group G, let $\mathcal{H}(G)$ denote the family of all Hausdorff group topologies on G, and $\mathcal{P}(G)$ the family of all precompact Hausdorff group topologies on G. Note that $\mathcal{H}(G)$ and $\mathcal{P}(G)$ are partially ordered sets with respect to set-theoretic inclusion of topologies.

Question 59. *Suppose that G and H are infinite Abelian groups. Must the groups G and H be (algebraically) isomorphic* 961–962?

(i) *if the posets $\mathcal{H}(G)$ and $\mathcal{H}(H)$ are isomorphic?*
(ii) *if the posets $\mathcal{P}(G)$ and $\mathcal{P}(H)$ are isomorphic?*

A relevant information (and the origin of this question) may be found in [**51**].

Acknowledgement. We thankfully acknowledge helpful comments on a preliminary version of this paper offered to us by M.J. Chasco, W. Comfort, K. Kunen, M. Megrelishvili, V. Pestov, E. Reznichenko and V. Uspenskij.

References

[1] A. V. Arhangel′skiĭ, *Topological homogeneity. Topological groups and their continuous images*, Uspekhi Mat. Nauk **42** (1987), no. 2(254), 69–105, Translation: Russian Math. Surveys 42 (1987) no. 2, 83–131.

[2] T. Banakh and L. Zdomskyy, *Closed locally path-connected subspaces of finite-dimensional topological groups are locally compact*, 2006, Preprint.

[3] M. M. Clementino and W. Tholen, *Tychonoff's theorem in a category*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3311–3314.

[4] W. W. Comfort, *Dense subgroups of compact groups*, Open problems in topology II (E. Pearl, ed.), Elsevier Science.

[5] W. W. Comfort, S. Hernández, D. Remus, and F. J. Trigos-Arrieta, *Some open questions on topological groups*, Nuclear groups and Lie groups (Madrid, 1999), Heldermann Verlag, Lemgo, 2001, pp. 57–76.

[6] W. W. Comfort, S. Hernández, and F. J. Trigos-Arrieta, *Cross sections and homeomorphism classes of abelian groups equipped with the Bohr topology*, Topology Appl. **115** (2001), no. 2, 215–233.

[7] W. W. Comfort and D. Remus, *Imposing pseudocompact group topologies on abelian groups*, Fund. Math. **142** (1993), no. 3, 221–240.

[8] W. W. Comfort and D. Remus, *Abelian torsion groups with a pseudocompact group topology*, Forum Math. **6** (1994), no. 3, 323–337.

[9] L. De Leo and D. Dikranjan, *The staightening theorem for bounded codomains*, Preprint.

[10] J. J. Dijkstra and J. van Mill, *Groups without convergent sequences*, Topology Appl. **74** (1996), 275–282.

[11] D. Dikranjan, *Compactness and connectedness in topological groups*, Topology Appl. **84** (1998), 227–252.

[12] D Dikranjan, *Continuous maps in the Bohr topology*, Appl. Gen. Topol. **2** (2001), no. 2, 237–270.

[13] D. Dikranjan, *A class of abelian groups defined by continuous cross sections in the Bohr topology*, Rocky Mountain J. Math. **32** (2002), no. 4, 1331–1355.

[14] D. Dikranjan, *Van Douwen's problems related to the Bohr topology*, Proc. Ninth Prague Topological Symposium (2001), Topology Atlas, Toronto, 2002, pp. 37–50.

[15] D. Dikranjan, I. R. Prodanov, and L. N. Stoyanov, *Topological groups: characters, dualities and minimal group topologies*, Marcel Dekker Inc., New York, 1990.

[16] D. Dikranjan and D. Shakhmatov, *Pseudocompact topologizations of groups*, Zb. Rad. (1990), no. 4, 83–93.

[17] D. Dikranjan and D. Shakhmatov, *Pseudocompact topologies on groups*, Topology Proc. **17** (1992), 335–342.

[18] D. Dikranjan and D. Shakhmatov, *Algebraic structure of pseudocompact groups*, Mem. Amer. Math. Soc. **133** (1998), no. 633, 83 pp.

[19] D. Dikranjan and D. Shakhmatov, *Compact-like group topologies on Abelian groups*, Contributed problems (Zoltán Balogh Memorial Topology Conference), 2002, `http://notch.mathstat.muohio.edu/balogh/`, pp. 6–9.

[20] D. Dikranjan and D. Shakhmatov, *Categorical aspects of the class of Abelian groups admitting a countably compact group topology*, 2003, Preprint.

[21] D. Dikranjan and D. Shakhmatov, *On the Markov–Zariski topology of an abelian group*, 2003, Preprint.

[22] D. Dikranjan and D. Shakhmatov, *Forcing hereditarily separable compact-like group topologies on abelian groups*, Topology Appl. **151** (2005), 2–54.

[23] D. Dikranjan and D. Shakhmatov, *Markov and Zariski topologies on the permutation group of integers*, 2006, Preprint.

[24] D. Dikranjan and M. G. Tkachenko, *Sequential completeness of quotient groups*, Bull. Austral. Math. Soc. **61** (2000), no. 1, 129–150.

[25] D. Dikranjan and M. G. Tkachenko, *Varieties generated by countably compact abelian groups*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2487–2496.

[26] D. Dikranjan and M. G. Tkachenko, *Algebraic structure of small countably compact abelian groups*, Forum Math. **15** (2003), no. 6, 811–837.

[27] D. Dikranjan, M. G. Tkachenko, and I. Yaschenko, *On transversal group topologies*, Topology Appl. **153** (2005), no. 5–6, 786–817.

[28] D. Dikranjan, M. G. Tkachenko, and I. Yaschenko, *Transversal group topologies on non-abelian groups*, Topology Appl. (2006), Article in press.

[29] D. Dikranjan and V. V. Uspenskij, *Categorically compact topological groups*, J. Pure Appl. Algebra **126** (1998), 149–168.

[30] D. Dikranjan and S. Watson, *A solution to van Douwen's problem on Bohr topologies*, J. Pure Appl. Algebra **163** (2001), no. 2, 147–158.

[31] E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), no. 2, 417–427.

[32] E. K. van Douwen, *The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G*, Topology Appl. **34** (1990), no. 1, 69–91.

[33] J. Galindo and S. García-Ferreira, *Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences*, Preprint.

[34] J. Galindo, S. Hernández, and T. S. Wu, *Recent results and open questions relating Chu duality and Bohr compactifications of locally compact groups*, Open problems in topology II (E. Pearl, ed.), Elsevier Science.

[35] J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), no. 2, 403–418.

[36] B. N. Givens, *Closed subsets which are not retracts in the Bohr topology*, Topology Appl. **129** (2003), no. 1, 11–14.

[37] B. N. Givens and K. Kunen, *Chromatic numbers and Bohr topologies*, Topology Appl. **131** (2003), no. 2, 189–202.

[38] H. Gladdines, *Countable closed sets that are not a retract of $G^{\#}$*, Topology Appl. **67** (1995), no. 2, 81–84.

[39] A. Hajnal and I. Juhász, *A separable normal topological group need not be Lindelöf*, General Topology and Appl. **6** (1976), no. 2, 199–205.

[40] J. E. Hart and K. Kunen, *Bohr compactifications of discrete structures*, Fund. Math. **160** (1999), no. 2, 101–151.

[41] K. Kozlov, E. Reznichenko, and O. Sipacheva, *Moscow questions on topological algebra*, Open problems in topology II (E. Pearl, ed.), Elsevier Science.

[42] J. Kulesza, *Spaces which do not embed in topological groups of the same dimension*, Topology Appl. **50** (1993), no. 2, 139–145.

[43] K. Kunen, *Bohr topologies and partition theorems for vector spaces*, Topology Appl. **90** (1998), 97–107.

[44] K. Kunen and W. Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 1, 117–137.

[45] G. Lukács, *Hereditarily h-complete groups*, Topology Appl. **145** (2004), 177–189.

[46] V. I. Malykhin, *An example of a topological group*, Topological spaces and their mappings, Latvijskij Gosudarstvennyj Universitet, Riga, 1981, pp. 120–123.

[47] V. I. Malykhin and L. B. Shapiro, *Pseudocompact groups without convergent sequences*, Mat. Zametki **37** (1985), no. 1, 103–109, Translation: Math. Notes 37 (1985) no. 1, 59–62.

[48] A. A. Markov, *On unconditionally closed sets*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **44** (1944), 180–181.

[49] J. van Mill, *Open problems in van Douwen's papers*, Collected papers. Vol. I, II, North-Holland, Amsterdam, 1994.

[50] B. A. Pasynkov, *The coincidence of various definitions of dimensionality for locally bicompact groups*, Dokl. Akad. Nauk SSSR **132** (1960), 1035–1037, Translation: Soviet Math. Dokl. 1 (1960) 720–722.

[51] D. Shakhmatov, *Character and pseudocharacter in minimal topological groups*, Mat. Zametki **38** (1985), no. 6, 908–914, Translation: Math. Notes 39 (1986) 465–470.

[52] D. Shakhmatov, *A problem of coincidence of dimensions in topological groups*, Topology Appl. **33** (1989), no. 1, 105–113.

[53] D. Shakhmatov, *Zerodimensionality of free topological groups and topological groups with noncoinciding dimensions*, Bull. Polish Acad. Sci. Math. **37** (1989), no. 7–12, 497–506.

[54] D. Shakhmatov, *Imbeddings into topological groups preserving dimensions*, Topology Appl. **36** (1990), no. 2, 181–204.

[55] D. Shakhmatov, *A survey of current researches and open problems in the dimension theory of topological groups*, Questions Answers Gen. Topology **8** (1990), no. 1, 101–128.

[56] D. Shakhmatov, *A note on transfinite inductive dimensions in topological groups*, Topology Appl. **54** (1993), 213–220.

[57] O. V. Sipacheva, *Consistent solution of Markov's problem about algebraic sets*, 2006, Preprint. `arXiv:math.GR/0605558`

[58] S. M. Sirota, *A product of topological groups, and extremal disconnectedness*, Mat. Sb. **79(121)** (1969), 179–192, Translation: Sb. Math. 8 (1969) 169–180.

[59] M. G. Tkachenko, *Countably compact and pseudocompact topologies on free abelian groups*, Izv. Vyssh. Uchebn. Zaved. Mat. (1990), no. 5, 68–75, Translation: Soviet Math. (Iz. VUZ) 34 (1990) no. 5, 79–86.

[60] M. G. Tkachenko, *Subgroups, quotient groups and products of $\mathbb{R}$-factorizable groups*, Topology Proc. **16** (1991), 201–231.

[61] M. G. Tkachenko, *Provisional solution to a Comfort–van Mill problem*, Topology Appl. **125** (2002), no. 1, 171–182.

[62] M. G. Tkachenko and I. Yaschenko, *Independent group topologies on abelian groups*, Topology Appl. **122** (2002), no. 1–2, 425–451.

[63] A. H. Tomita, *Two countably compact topological groups: one of size $\aleph_\omega$ and the other of weight $\aleph_\omega$ without non-trivial convergent sequences*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2617–2622.