EXAMPLES CONCERNING EXTENSIONS OF CONTINUOUS FUNCTIONS

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Abstract. Given a space Y, let us say that a space X is a *total extender for* Y provided that every continuous map $f\colon A \to Y$ defined on a subspace A of X admits a continuous extension $\tilde{f} : X \to Y$ over X. The first author and Alberto Marcone proved that a space X is hereditarily extremally disconnected and hereditarily normal if and only if it is a total extender for every compact metrizable space Y, and asked whether the same result holds without any assumption of metrizability on Y. We demonstrate that a hereditarily extremally disconnected, hereditarily normal, non-collectionwise Hausdorff space X constructed by K. Kunen is not a total extender for K, the one-point compactification of the discrete space of size ω_1. Under the assumption $2^{\omega_0} = 2^{\omega_1}$, we provide an example of a separable, hereditarily extremally disconnected, hereditarily normal space X that is not a total extender for K. Furthermore, using forcing we prove that, in the generic extension of a model of ZFC + MA(ω_1), every first-countable separable space X of size ω_1 has a finer topology τ on X such that (X, τ) is still separable and fails to be a total extender for K. We also show that a hereditarily extremally disconnected, hereditarily separable space X satisfying some stronger form of hereditary normality (so-called structural normality) is a total extender for every compact Hausdorff space, and we give a non-trivial example of such an X.

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0. Introduction and preliminary results.

In the paper [CM] it is proved that a topological space X is (hereditarily) normal and hereditarily extremally disconnected if and only if for every compact metrizable space Y and every continuous function f from a subspace of X to Y, there is a continuous $\tilde{f}\colon X \to Y$ which extends f—Theorem 7.5. In the same paper, the authors show that the assumption of compactness on Y cannot be dropped; actually, this follows from a still more negative result (see Proposition 7.7).

Of course, another natural question in this vein—which is pointed out as well by the authors of [CM]—would be whether in the above statement we may drop the hypothesis of metrizability on Y (leaving the assumption of compactness unchanged). In this paper we first give a negative answer to such a question, using as

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X a space constructed by K. Kunen in 1977. Then, a new question which naturally arises is whether we may further require, in the counterexample, the space X to be separable. Actually, we do not know what the answer is in ZFC; however, we show that it is possible to obtain such a separable X by assuming that $2^{\omega_0} = 2^{\omega_1}$. Finally (and this is, in some sense, the core of the paper), we use a suitable notion of forcing to show that, consistently, it is possible to adopt a similar procedure to get our desired X, starting from any space belonging to a rather large class—namely, the separable first-countable spaces of cardinality ω_1.

The last section of the paper is devoted to the study of some (nontrivial) situations where the extension property works. It turns out that a very strong version of normality introduced in [CM] (namely, *structural normality*), together with hereditary separability, may play a crucial rôle in obtaining spaces X having the extension property with respect to every compact space Y (cf. Corollary 11).

All the counterexamples to the extension property that we provide in this paper are founded on the same basic idea, that is illustrated by the following proposition (to be systematically used in the next sections). Such a result should also make clear why it is so natural to wonder about the existence of a *separable* counterexample, once we have one of density ω_1; actually, the question seems to be linked with some combinatorial properties of $\wp(\omega)$.

Proposition 1. *Let E', E'' be two disjoint (infinite) sets and suppose to have associated to every $x \in E'$ a non-principal ultrafilter $\mathcal{U}(x)$ on E'', in such a way that the following hold:*

1) for every $L \subseteq E'$, there is a function U on E' such that $U(x) \in \mathcal{U}(x)$ for every $x \in E'$, and

$$\Big(\bigcup_{x\in L} U(x)\Big) \cap \Big(\bigcup_{x\in E'\setminus L} U(x)\Big) = \emptyset;$$

2) there is no function U associating to every $x \in E'$ a $U(x) \in \mathcal{U}(x)$, such that $U(x) \cap U(y) = \emptyset$ for all distinct $x, y \in E'$.

Put $X = E' \cup E''$, and let τ be the topology on X making all elements of E'' isolated, while every $x \in E'$ has a fundamental system of (open) τ-neighborhoods given by: $\big\{\{x\} \cup U \,\big|\, U \in \mathcal{U}(x)\big\}$. Then (X, τ) is a (hereditarily) normal, hereditarily extremally disconnected space; moreover, if $|E'| = \zeta$ and Y is any compactification of $D(\zeta)$ (=the discrete space of cardinality ζ, in accordance with [En, Example 1.4.20]*), then every one-to-one mapping f from E' onto $D(\zeta)$ turns out to be a continuous function from a subspace of (X, τ) to Y which cannot be extended to any continuous $\tilde{f}\colon (X, \tau) \to Y$.*

Proof.

a) Normality. Of course, X is T_1 because the ultrafilters $\mathcal{U}(x)$ are non-principal. Let C_1, C_2 be closed disjoint subsets of X: Then $A_i = C_i \cap E''$ is open in X for $i = 1, 2$. Let also $L_i = C_i \cap E'$ for $i = 1, 2$: As a consequence of 1), it is possible to associate to every $x \in L_1 \cup L_2$ a $U(x) \in \mathcal{U}(x)$ in such a way that

$$\Big(\bigcup_{x\in L_1} U(x)\Big) \cap \Big(\bigcup_{x\in L_2} U(x)\Big) = \emptyset.$$

Then it is easily seen that

$$W_1 = \big(A_1 \cup \bigcup_{x\in L_1} (\{x\} \cup U(x))\big) \cap (X \setminus C_2)$$

and

$$W_2 = \big(A_2 \cup \bigcup_{x \in L_2} (\{x\} \cup U(x))\big) \cap (X \setminus C_1)$$

are disjoint open sets which include C_1 and C_2, respectively.

b) Hereditary extremal disconnectedness. We will use the following criterion (cf. [KP, Theorem 2] or [CM, Lemma 7.3]): A T_2-space is hereditary extremally disconnected if and only if for every pair A, B of its subsets with $\overline{A} \cap B = \overline{B} \cap A = \emptyset$ (i.e., for every A, B *separated* subsets of X), we have that $\overline{A} \cap \overline{B} = \emptyset$. Thus, suppose $A, B \subseteq X$ are separated: We only have to show that no element of E' may be adherent to both A and B. Actually, given $\bar{x} \in E'$, we may assume that $\bar{x} \notin A \cup B$ (otherwise, since $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, we would have either $\bar{x} \notin \overline{B}$ or $\bar{x} \notin \overline{A}$). As $\mathcal{U}(\bar{x})$ is an ultrafilter on E'', it must contain either $A \cap E''$ or $E'' \setminus (A \cap E'')$; therefore, either $\{\bar{x}\} \cup (A \cap E'')$ is a neighborhood of $\bar{x}$ disjoint from B, or $\{\bar{x}\} \cup \big(E'' \setminus (A \cap E'')\big)$ is a neighborhood of $\bar{x}$ disjoint from A.

c) Non-extensibility. Suppose that $\tilde{f}$ is a continuous extension of f. Putting, for every $x \in E'$, $V_x = \tilde{f}^{-1}\big(\{f(x)\}\big)\big(= \tilde{f}^{-1}(\{\tilde{f}(x)\})\big)$ would give pairwise disjoint neighborhoods of the points x. Then, intersecting with E'', we would contradict (2). □

Remark. If E', E'' are such that $|E'| > |E''|$, then condition 2) of the statement of the above proposition is automatically satisfied.

Remark. Using standard techniques about C^*-embedded spaces, it is easy to prove (cf., for example, [CM, Proposition 7.4], which partially relies on [GJ, Exercise 6R2]) that for a normal, extremally disconnected space, hereditary normality is equivalent to hereditary disconnectedness. For a measure-theoretic proof of the same result, see [He, Theorem 2.8].

Remark. If, in the above proposition, we take as Y the one-point compactification of $D(\zeta)$, then Y is a Fréchet-Urysohn, Eberlein compact space. Moreover, Y is also α_1 (for the definition, see for example [No, §2]). As a consequence, the spaces X from Examples 4, 5 and Corollary 8 have the non-extension property even with respect to some suitable compact space Y with the above additional characteristics.

1. The ZFC and ZFC+($2^{\omega_0} = 2^{\omega_1}$) examples.

In both constructions produced in this section, a basic rôle will be played by the classical notion of *independent family*. Recall that a family $\{A_i\}_{i \in I}$ of subsets of a set M is said to be an independent family if for every pair I_1, I_2 of finite disjoint subsets of I, the set $\big(\bigcap_{i \in I_1} A_i\big) \setminus \big(\bigcup_{i \in I_2} A_i\big)$ is nonempty. The two lemmas below state some very natural links between independent families and finitely additive measures.

Actually, Lemma 2 is well-known enough as folklore among experts of abstract measure theory; but a precise reference for it seems quite hard to find in the literature (cf., [Gr] and [Ba]). Thus, for the sake of completeness, we have given it a proof.

Lemma 2. *Let $\{A_i\}_{i \in I}$ be an infinite independent family on a set M. Then there exists a finitely additive measure μ on $\wp(M)$ such that for every finite $J \subseteq I$ and* *every $s\colon J \to 2$, we have that*

$$\mu\Big(\bigcap_{i\in J} A_i^{s(i)}\Big) = \frac{1}{2^{|J|}}, \tag{Δ}$$

where $A_i^1 = A_i$ and $A_i^0 = M \setminus A_i$ for every $i \in I$.

Proof. Let $\mathcal{A}'$ be the collection of all sets $\bigcap_{i\in J} A_i^{s(i)}$, with $J \in [I]^{<\omega}$ and $s\colon J \to 2$ (so that, in particular, $M = \bigcap_{i\in\emptyset} A_i^{\langle\rangle(i)} \in \mathcal{A}'$), and $\mathcal{A}$ be the collection of all finite unions of elements of $\mathcal{A}'$. Then $\mathcal{A}$ is an algebra of sets, as it is closed under finite union and complementation—to see the last fact, consider that

$$\begin{aligned} M \setminus \bigcup_{h=1}^{n} \big(\bigcap_{i\in J_h} A_i^{\vartheta_h(i)}\big) &= \bigcap_{h=1}^{n} \big(\bigcup_{i\in J_h} (M \setminus A_i^{\vartheta_h(i)})\big) \\ &= \bigcap_{h=1}^{n} \big(\bigcup_{i\in J_h} A_i^{1-\vartheta_h(i)}\big) = \bigcup_{(i_1,\ldots,i_n)\in J_1\times\ldots\times J_n} \big(\bigcap_{h=1}^{n} A_{i_h}^{1-\vartheta_h(i_h)}\big). \end{aligned}$$

If we can prove the existence of a finitely additive measure μ on $\mathcal{A}$ satisfying (Δ), then a well-known extension property (see, for example, [Wa, Theorem 10.7] or [Fr2, Corollary 391G]) will give the desired result.

Let $\mathrm{Fn}(I,2)$ be the set of all functions from a finite subset of I to 2, and

$$\tilde{\mathcal{A}} = \Big\{\Theta \in [\mathrm{Fn}(I,2)]^{<\omega} \,\Big|\, \Big(\bigcap_{i\in\operatorname{dom}\vartheta'} A_i^{\vartheta'(i)}\Big) \cap \Big(\bigcap_{i\in\operatorname{dom}\vartheta''} A_i^{\vartheta''(i)}\Big) = \emptyset \quad \text{for distinct } \vartheta',\vartheta'' \in \Theta\Big\};$$

then, for every $\Theta \in \tilde{\mathcal{A}}$, put

$$\tilde{A}(\Theta) = \bigcup_{\vartheta\in\Theta} \Big(\bigcap_{i\in\operatorname{dom}\vartheta} A_i^{\vartheta(i)}\Big) \;(\in \mathcal{A}) \quad \text{and} \quad \tilde{\mu}(\Theta) = \Sigma_{\vartheta\in\Theta} \frac{1}{2^{|\operatorname{dom}\vartheta|}}.$$

Observe that $\tilde{A}\colon \tilde{\mathcal{A}} \to \mathcal{A}$ is onto. Indeed, suppose that $A = \bigcup_{h=1}^{n} \big(\bigcap_{i\in J_h} (A_i^{\vartheta_h(i)})\big)$ is an element of $\mathcal{A}$, and put $J = \bigcup_{h=1}^{n} J_h$: For every $h \in \{1,\ldots,n\}$, letting

$$\Theta_h = \big\{\vartheta \in {}^J2 \,\big|\, \vartheta\restriction_{J_h} = \vartheta_h\big\},$$

we easily see that $\bigcap_{i\in J_h} A_i^{\vartheta_h(i)} = \bigcup_{\vartheta\in\Theta_h} \big(\bigcap_{i\in J} A_i^{\vartheta(i)}\big)$. Therefore, putting $\Theta = \bigcup_{h=1}^{n} \Theta_h$, we obtain that $A = \bigcup_{\vartheta\in\Theta} \big(\bigcap_{i\in\operatorname{dom}\vartheta} A_i^{\vartheta(i)}\big) = \tilde{A}(\Theta)$ (observe that Θ belongs to $\tilde{\mathcal{A}}$ because all its elements have the same domain J, so that for distinct $\vartheta',\vartheta'' \in \Theta$ there must be $\hat{\imath} \in J$ with $\vartheta'(\hat{\imath}) \neq \vartheta''(\hat{\imath})$, and hence $\big(\bigcap_{i\in J} A_i^{\vartheta'(i)}\big) \cap \big(\bigcap_{i\in J} A_i^{\vartheta''(i)}\big) \subseteq A_{\hat{\imath}}^{\vartheta'(\hat{\imath})} \cap A_{\hat{\imath}}^{\vartheta''(\hat{\imath})} = \emptyset$).

Now we prove that, given any $\Theta',\Theta'' \in \tilde{\mathcal{A}}$ such that $\tilde{A}(\Theta') = \tilde{A}(\Theta'')$, we have $\tilde{\mu}(\Theta') = \tilde{\mu}(\Theta'')$. Indeed, if we are in such a situation, we put $J = \bigcup_{\vartheta\in\Theta'\cup\Theta''} \operatorname{dom}\vartheta$ and, for every $\vartheta \in \Theta' \cup \Theta''$, $\Lambda_\vartheta = \big\{\lambda \in {}^J2 \,\big|\, \lambda\restriction_{\operatorname{dom}\vartheta} = \vartheta\big\}$. Then $\bigcap_{i\in\operatorname{dom}\vartheta} A_i^{\vartheta(i)} =$

$\bigcup_{\lambda\in\Lambda_\vartheta}\big(\bigcap_{i\in J} A_i^{\lambda(i)}\big)$ for every $\vartheta\in\Theta'\cup\Theta''$; moreover, letting $\Lambda'=\bigcup_{\vartheta\in\Theta'}\Lambda_\vartheta$ and $\Lambda''=\bigcup_{\vartheta\in\Theta''}\Lambda_\vartheta$, we have (like before) that $\Lambda',\Lambda''\in\tilde{\mathcal{A}}$, and that

$$\begin{aligned}\tilde{A}(\Lambda') &= \bigcup_{\lambda\in\Lambda'}\Big(\bigcap_{i\in\operatorname{dom}\lambda} A_i^{\lambda(i)}\Big) = \bigcup_{\vartheta\in\Theta'}\bigcup_{\lambda\in\Lambda_\vartheta}\Big(\bigcap_{i\in\operatorname{dom}\lambda} A_i^{\lambda(i)}\Big)\\ &= \bigcup_{\vartheta\in\Theta'}\Big(\bigcap_{i\in\operatorname{dom}\vartheta} A_i^{\vartheta(i)}\Big) = \tilde{A}(\Theta') = \tilde{A}(\Theta'') = \bigcup_{\vartheta\in\Theta'}\Big(\bigcap_{i\in\operatorname{dom}\vartheta} A_i^{\vartheta(i)}\Big)\\ &= \bigcup_{\vartheta\in\Theta''}\bigcup_{\lambda\in\Lambda_\vartheta}\Big(\bigcap_{i\in\operatorname{dom}\lambda} A_i^{\lambda(i)}\Big) = \bigcup_{\lambda\in\Lambda''}\Big(\bigcap_{i\in\operatorname{dom}\lambda} A_i^{\lambda(i)}\Big) = \tilde{A}(\Lambda'').\end{aligned}$$

Since all the elements of $\Lambda'\cup\Lambda''$ have the same domain, and the family $\{A_i\}_{i\in I}$ is independent, from $\tilde{A}(\Lambda')=\tilde{A}(\Lambda'')$ we easily deduce that $\Lambda'=\Lambda''$. Also, we have that $\tilde{\mu}(\Lambda')=\Sigma_{\lambda\in\Lambda'}\frac{1}{2^{|\operatorname{dom}\lambda|}}=|\Lambda'|\cdot\frac{1}{2^{|J|}}=\frac{1}{2^{|J|}}\Sigma_{\vartheta\in\Theta'}|\Lambda_\vartheta|=\frac{1}{2^{|J|}}\Sigma_{\vartheta\in\Theta'}2^{|J|-|\operatorname{dom}\vartheta|}=\Sigma_{\vartheta\in\Theta'}\frac{1}{2^{|\operatorname{dom}\vartheta|}}=\tilde{\mu}(\Theta')$, and in a symmetric way it is proved that $\tilde{\mu}(\Lambda'')=\tilde{\mu}(\Theta'')$. Therefore, $\tilde{\mu}(\Theta')=\tilde{\mu}(\Lambda')=\tilde{\mu}(\Lambda'')=\tilde{\mu}(\Theta'')$.

The properties of $\tilde{A}$ and $\tilde{\mu}$ we have proved so far allow us to define $\mu\colon\mathcal{A}\to[0,+\infty[$ (actually, $\mu\colon\mathcal{A}\to[0,1]$) by $\mu(A)=\tilde{\mu}(\Theta)$, where $\Theta\in\tilde{\mathcal{A}}$ is such that $\tilde{A}(\Theta)=A$. Then it is easy to see that μ satisfies condition (Δ) of the statement—observe, in particular, that taking as s the empty function we have $\mu(M)=\mu\big(\bigcap_{i\in\emptyset}A_i^{s(i)}\big)=\tilde{\mu}(\{\emptyset\})=\frac{1}{2^0}=1$. Suppose now to have two disjoint $A_1,A_2\in\mathcal{A}$, with $A_1=\tilde{A}(\Theta_1)$ and $A_2=\tilde{A}(\Theta_2)$: then $\big(\bigcap_{i\in\operatorname{dom}\vartheta}A_i^{\vartheta(i)}\big)\cap\big(\bigcap_{i\in\operatorname{dom}\lambda}A_i^{\lambda(i)}\big)=\emptyset$ for different ϑ,λ belonging either both to Θ_1 or both to Θ_2. Moreover, since $A_1\cap A_2=\emptyset$, the above intersection is empty also if $\vartheta\in\Theta_1$ and $\lambda\in\Theta_2$—or vice-versa. Therefore, $\Theta_1\cap\Theta_2=\emptyset$, $\Theta_1\cup\Theta_2\in\tilde{\mathcal{A}}$, $\tilde{A}(\Theta_1\cup\Theta_2)=A_1\cup A_2$, and $\mu(A_1\cup A_2)=\tilde{\mu}(\Theta_1\cup\Theta_2)=\Sigma_{\vartheta\in\Theta_1\cup\Theta_2}\frac{1}{2^{|\operatorname{dom}\vartheta|}}=\big(\Sigma_{\vartheta\in\Theta_1}\frac{1}{2^{|\operatorname{dom}\vartheta|}}\big)+\big(\Sigma_{\vartheta\in\Theta_2}\frac{1}{2^{|\operatorname{dom}\vartheta|}}\big)=\tilde{\mu}(\Theta_1)+\tilde{\mu}(\Theta_2)=\mu(A_1)+\mu(A_2)$. ☐

Lemma 3. *Let $\{A_i\}_{i\in I}$ be an (infinite) independent family on a set M, and μ a finitely additive measure on $\wp(M)$ satisfying condition (Δ) of Lemma 2. Then for every $\varphi\colon I\to 2$ there is an ultrafilter $\mathcal{U}$ on M such that $\{A_i^{\varphi(i)}\mid i\in I\}\subseteq\mathcal{U}$ and $\forall F\in\mathcal{U}\colon\mu(F)>0$.*

Proof. The collection

$$\Phi=\big\{\mathcal{F}\,\big|\,\mathcal{F}\text{ is a filter on }M,\ \{A_i^{\varphi(i)}\mid i\in I\}\subseteq\mathcal{F},\ \forall F\in\mathcal{F}\colon\mu(F)>0\big\}$$

is clearly inductive; also, it is nonempty, because by condition (Δ) of Lemma 2 and by monotonicity of μ it contains the filter generated by $\{A_i^{\varphi(i)}\mid i\in I\}$. Thus, by Zorn's lemma, Φ contains a maximal element $\mathcal{U}$. By contradiction, suppose $\mathcal{U}$ is not an ultrafilter: Then there is $L\subseteq M$ such that $L,M\setminus L\notin\mathcal{U}$. Notice that either $\forall F\in\mathcal{U}\colon\mu(F\cap L)>0$ or $\forall F\in\mathcal{U}\colon\mu\big(F\cap(M\setminus L)\big)>0$ (or both): Otherwise, there would be $F_1,F_2\in\mathcal{U}$ with $\mu(F_1\cap L)=\mu\big(F_2\cap(M\setminus L)\big)=0$, so that by monotonicity $\mu\big((F_1\cap F_2)\cap L\big)=\mu\big((F_1\cap F_2)\cap(M\setminus L)\big)=0$, and hence $\mu(F_1\cap F_2)=\mu\big((F_1\cap F_2)\cap L\big)+\mu\big((F_1\cap F_2)\cap(M\setminus L)\big)=0+0=0$, which would contradict $F_1\cap F_2\in\mathcal{U}\in\Phi$. Therefore, either $\{F\cap L\mid F\in\mathcal{U}\}$ or $\{F\cap(M\setminus L)\mid F\in\mathcal{U}\}$ is a basis for a filter on M which properly extends $\mathcal{U}$ and contains no zero-measure set. A contradiction. ☐

Remark. The ultrafilter $\mathcal{U}$ provided by the above lemma is always non-principal, as $\mu(\{x\}) = 0$ for every $x \in M$. Indeed, if by contradiction $\mu(\{x\}) > \frac{1}{2^{n^*}}$ for some $n^* \in \omega$, then fix any $J \subseteq I$ with $|J| = n^*$ and let $L = \left(\bigcap_{i\in J_1} A_i\right) \setminus \left(\bigcup_{i\in J_2}(M \setminus A_i)\right)$, where $J_1 = \{i \in J \mid x \in A_i\}$ and $J_2 = \{i \in J \mid x \notin A_i\}$. Then $\mu(L) = \frac{1}{2^{n^*}}$ and $x \in L$, which contradicts the monotonicity of μ.

A basic fact about independent families, that we will use in the next examples, is the well-known Hausdorff-Fichtenholz-Kantorovich theorem, which claims that on any set M of cardinality κ there is an independent family $\{A_i\}_{i\in I}$ with $|I| = 2^\kappa$. This may be proved in a topological fashion, considering the traces of basic open subsets of $\{0,1\}^{2^\kappa}$ on a dense subset of cardinality κ. A different, purely combinatorial proof is sketched in [Ku1, Exercise VIII.(A7)].

The first example we are going to illustrate comes from [Ku2], where the statements of Lemmas 2 and 3 are also implicit.

Example 4. *There are sets E', E'', both of cardinality ω_1, such that conditions 1),2) of Proposition 1 are satisfied.*

Proof. Let E', E'' be disjoint sets with $|E'| = |E''| = \omega_1$, and let $\mathcal{F} = \{A_\alpha\}_{\alpha\in 2^{\omega_1}}$ be an independent family on E''. By Lemma 2, there is a finitely additive measure μ on $\wp(E'')$ such that condition (Δ) is satisfied.

Now, let $\{H_\alpha\}_{\alpha\in 2^{\omega_1}}$ list $\wp(E')$, and for every $x \in E'$ define $\varphi_x\colon 2^{\omega_1} \to 2$ by:

$$\varphi_x(\alpha) = \begin{cases} 1 & \text{if } x \in H_\alpha; \\ 0 & \text{if } x \notin H_\alpha. \end{cases}$$

Then by Lemma 3 and the subsequent remark, we may associate to every $x \in E'$ a non-principal ultrafilter $\mathcal{U}(x)$ on E'' such that

$$\{A_\alpha^{\varphi_x(\alpha)} \mid \alpha \in 2^{\omega_1}\} \subseteq \mathcal{U}(x) \quad \text{and} \quad \forall F \in \mathcal{U}(x)\colon \mu(F) > 0.$$

Let us show that conditions 1),2) of Proposition 1 are fulfilled. If L is any subset of E', then $L = H_{\hat\alpha}$ for some $\hat\alpha \in 2^{\omega_1}$, and we have $\varphi_x(\hat\alpha) = 1$ for every $x \in L$ and $\varphi_x(\hat\alpha) = 0$ for every $x \in E' \setminus L$. Therefore, putting $U(x) = A_{\hat\alpha}$ for every $x \in L$ and $U(x) = E'' \setminus A_{\hat\alpha}$ for every $x \in E' \setminus L$, we have that $U(x) \in \mathcal{U}(x)$ for every $x \in E'$ and that $\left(\bigcup_{x\in L} U(x)\right) \cap \left(\bigcup_{x\in E'\setminus L} U(x)\right) = A_{\hat\alpha} \cap (E'' \setminus A_{\hat\alpha}) = \emptyset$.

To prove condition 2) of Proposition 1, consider an arbitrary function U which associates to every $x \in E'$ a $U(x) \in \mathcal{U}(x)$. Since $\mu\big(U(x)\big) > 0$ for every $x \in E'$, there are $n^* \in \omega$ and an infinite $M' \subseteq E'$ such that $\mu\big(U(x)\big) \geq \frac{1}{n^*}$ for every $x \in M'$. Then it is impossible that the family $\{U(x)\}_{x\in E'}$ consist of pairwise disjoint sets, because taking a finite $F \subseteq M'$ with $|F| > n^*$, and using finite additivity and monotonicity, we would get a contradiction with $\mu(E'') = 1$. $\square$

Remark. The construction of the space, having essentially the same properties of the one outlined by K. Kunen in [Ku2], that has been carried out in [Ta, Example D] involves a *maximal* independent family and the Δ-system lemma, thereby avoiding any recourse to finitely additive measures. As pointed out by P.J. Nyikos in his review of [Ta] for *Mathematical Reviews*, this argument has a gap. P.J. Nyikos suggests to fulfill the gap by using an independent family of sets whose traces form a maximal independent family on every intersection of a finite subfamily and the complements of another, disjoint, finite subfamily. Even if this modification would work, we have preferred to give a self-contained argument traced back to Kunen's original outline [Ku2]. Incidentally, the proof of normality of X in [Ta, Example D] also has a minor gap (that is quite easy to fix).

Example 5 ($2^{\omega_0} = 2^{\omega_1}$)**.** *There are sets E', E'', with $|E'| = \omega_1$ and $|E''| = \omega$, such that conditions 1),2) of Proposition 1 are satisfied.*

Proof. Let E', E'' be disjoint sets having the required cardinalities. Fix an independent family $\{A_\alpha\}_{\alpha\in 2^\omega}$ on E'' and, using $2^{\omega_0} = 2^{\omega_1}$, list $\wp(E')$ as $\{H_\alpha\}_{\alpha\in 2^\omega}$. Now define the functions φ_x an in Example 4, and let each non-principal ultrafilter $\mathcal{U}(x)$ include the collection $\{A_\alpha^{\varphi_x(\alpha)} \mid \alpha \in 2^{\omega_1}\}$ (without any other restriction). Then condition 1) of Proposition 1 is proved as in Example 4, and condition 2) is immediate when taking into account the first remark after the same proposition. □

2. The forcing construction.

Lemma 6. *If κ is a cardinal for which $MA(\kappa)$ holds, I, J are sets (of indices) both of cardinality $\leq \kappa$ and, for every $i \in I$ and $j \in J$, A_i and B_j are elements of $[\omega]^\omega$, then it is possible to associate to every $i \in I$ and $j \in J$ sets $A'_i, B'_j \in [\omega]^\omega$, so that:*

$$\forall i \in I\colon \forall j \in J\colon (A'_i \subseteq A_i \ \wedge \ B'_j \subseteq B_j \ \wedge \ A'_i \cap B'_j = \emptyset).$$

Proof. Let $\mathbb{P}$ be the set of all pairs (φ, ψ), where φ, ψ are functions, $\operatorname{dom}\varphi \in [I]^{<\omega}$, $\operatorname{dom}\psi \in [J]^{<\omega}$, and:

1) $\forall i \in \operatorname{dom}\varphi\colon \varphi(i) \in [A_i]^{<\omega}$;
2) $\forall j \in \operatorname{dom}\psi\colon \psi(j) \in [B_j]^{<\omega}$;
3) $\left(\bigcup_{i\in\operatorname{dom}\varphi} \varphi(i)\right) \cap \left(\bigcup_{j\in\operatorname{dom}\psi} \psi(j)\right) = \emptyset$.

For $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathbb{P}$, let:

$$(\varphi_1, \psi_1) \geq (\varphi_2, \psi_2) \Longleftrightarrow \big(\operatorname{dom}\varphi_1 \subseteq \operatorname{dom}\varphi_2 \ \wedge \ (\forall i \in \operatorname{dom}\varphi_1\colon \varphi_1(i) \subseteq \varphi_2(i)) \\ \wedge \ \operatorname{dom}\psi_1 \subseteq \operatorname{dom}\psi_2 \ \wedge \ (\forall i \in \operatorname{dom}\psi_1\colon \psi_1(i) \subseteq \psi_2(i))\big)$$

(which intuitively means, as usual, that (φ_2, ψ_2) *extends* (φ_1, ψ_1)). Observe that two $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ in $\mathbb{P}$ are compatible, i.e. have a common extension, if and only if

$$\Big(\bigcup_{i\in\operatorname{dom}\varphi_1} \varphi_1(i)\Big) \cap \Big(\bigcup_{j\in\operatorname{dom}\psi_2} \psi_2(j)\Big) = \emptyset$$

and

$$\Big(\bigcup_{i\in\operatorname{dom}\varphi_2} \varphi_2(i)\Big) \cap \Big(\bigcup_{j\in\operatorname{dom}\psi_1} \psi_1(j)\Big) = \emptyset.$$

In this case, a common extension is given by $(\varphi_1 \underline{\cup} \varphi_2, \psi_1 \underline{\cup} \psi_2)$, where for any two functions f, g, the function $f \underline{\cup} g$ is such that $\operatorname{dom}(f \underline{\cup} g) = \operatorname{dom} f \cup \operatorname{dom} g$, and for x in this set we have:

$$(f \underline{\cup} g)(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f \setminus \operatorname{dom} g; \\ g(x) & \text{if } x \in \operatorname{dom} g \setminus \operatorname{dom} f; \\ f(x) \cup g(x) & \text{if } x \in \operatorname{dom} f \cap \operatorname{dom} g. \end{cases}$$

First of all, we prove that $\mathbb{P}$ is c.c.c. Observe that if $(\varphi_1,\psi_1),(\varphi_2,\psi_2)\in\mathbb{P}$ are such that $\bigcup_{i\in\operatorname{dom}\varphi_1}\varphi_1(i)=\bigcup_{i\in\operatorname{dom}\varphi_2}\varphi_2(i)$ and $\bigcup_{j\in\operatorname{dom}\psi_1}\psi_1(j)=\bigcup_{j\in\operatorname{dom}\psi_2}\psi_2(j)$, then they are certainly compatible. Since for every $(\varphi,\psi)\in\mathbb{P}$, the sets

$$\bigcup_{i\in\operatorname{dom}\varphi_1}\varphi_1(i)\quad\text{and}\quad\bigcup_{j\in\operatorname{dom}\psi_1}\psi_1(j)$$

are both in $[\omega]^{<\omega}$, and $\left|[\omega]^{<\omega}\times[\omega]^{<\omega}\right|=\omega$, in every $S\subseteq\mathbb{P}$ with $|S|>\omega$ there must be two distinct elements $(\varphi_1,\psi_1),(\varphi_2,\psi_2)$ with $\bigcup_{i\in\operatorname{dom}\varphi_1}\varphi_1(i)=\bigcup_{i\in\operatorname{dom}\varphi_2}\varphi_2(i)$ and $\bigcup_{j\in\operatorname{dom}\psi_1}\psi_1(j)=\bigcup_{j\in\operatorname{dom}\psi_2}\psi_2(j)$, so that S is not an antichain.

For every $i\in I$, $j\in J$ and $n\in\omega$, let

$$D'_{i,n}=\left\{(\varphi,\psi)\in\mathbb{P}\,\middle|\,i\in\operatorname{dom}\varphi\ \wedge\ |\varphi(i)|>n\right\}$$

and

$$D''_{j,n}=\left\{(\varphi,\psi)\in\mathbb{P}\,\middle|\,j\in\operatorname{dom}\psi\ \wedge\ |\psi(j)|>n\right\};$$

we claim that both of them are dense in $\mathbb{P}$. To prove the first fact (the second one is symmetric), let (φ,ψ) be any element of $\mathbb{P}$, and chose n distinct elements $m_1,\dots,m_n\in A_i\setminus\left(\bigcup_{j\in\operatorname{dom}\psi}\psi(j)\right)$; then define φ^* to be the function having as domain: $\operatorname{dom}\varphi\cup\{i\}$, and such that $\varphi^*(i')=\varphi(i')$ for $i'\in\operatorname{dom}\varphi\setminus\{i\}$, and

$$\varphi^*(i)=\begin{cases}\{m_1,\dots,m_n\} & \text{if } i\notin\operatorname{dom}\varphi;\\ \varphi(i)\cup\{m_1,\dots,m_n\} & \text{if } i\in\operatorname{dom}\varphi.\end{cases}$$

It is clear that, in any case, $(\varphi^*,\psi)\in\mathbb{P}$, $(\varphi^*,\psi)\le(\varphi,\psi)$, $i\in\operatorname{dom}\varphi^*$ and $|\varphi^*(i)|\ge n$.

Now, by MA(κ), there exists a filter G on $\mathbb{P}$ which meets every $D'_{i,n}$ and $D''_{j,n}$, for $i\in I$, $j\in J$ and $n\in\omega$. For every $i\in I$ and $j\in J$, let

$$A'_i=\bigcup\{\varphi(i)\mid(\exists\psi\colon(\varphi,\psi)\in G)\ \wedge\ i\in\operatorname{dom}\varphi\}$$

and

$$B'_j=\bigcup\{\psi(j)\mid(\exists\varphi\colon(\varphi,\psi)\in G)\ \wedge\ j\in\operatorname{dom}\psi\};$$

it is clear that $A'_i\subseteq A_i$ and $B'_j\subseteq B_j$. Moreover, all the A'_i and B'_j are infinite: Indeed, for every $n\in\omega$, there is an element $(\tilde\varphi,\tilde\psi)\in G\cap D'_{i,n}$ $[(\tilde\varphi,\tilde\psi)\in G\cap D''_{j,n}]$. Then $i\in\operatorname{dom}\tilde\varphi$ and $|\tilde\varphi(i)|\ge n$ $[j\in\operatorname{dom}\tilde\psi$ and $|\tilde\psi(j)|\ge n]$, and hence

$$|A'_i|=\left|\bigcup\{\varphi(i)\mid(\exists\psi\colon(\varphi,\psi)\in G)\ \wedge\ i\in\operatorname{dom}\varphi\}\right|\ge\left|\tilde\varphi(i)\right|\ge n$$

$$\left[\,|B'_j|=\left|\bigcup\{\psi(j)\mid(\exists\varphi\colon(\varphi,\psi)\in G)\ \wedge\ j\in\operatorname{dom}\psi\}\right|\ge|\tilde\psi(j)|\ge n\,\right].$$

Finally, we prove that $\left(\bigcup_{i\in I}A'_i\right)\cap\left(\bigcup_{j\in J}B'_j\right)=\emptyset$. By contradiction, suppose there are $i\in I$, $j\in J$ and $\bar m\in\omega$ such that $\bar m\in A'_i\cap B'_j$. Then $\bar m\in\varphi_1(i)$ for some $(\varphi_1,\psi_1)\in G$, and $\bar m\in\psi_2(j)$ for some $(\varphi_2,\psi_2)\in G$. Let $(\tilde\varphi,\tilde\psi)\in G$ be a common extension of (φ_1,ψ_1) and (φ_2,ψ_2): Then $\bar m\in\varphi_1(i)\subseteq\tilde\varphi(i)$ and $\bar m\in\psi_2(j)\subseteq\tilde\psi(j)$, which is impossible because $(\tilde\varphi,\tilde\psi)\in\mathbb{P}$. $\square$

Theorem 7. *Let M be a countable transitive model of ZFC + MA(ω_1). Let, in M, $\mathfrak{K}$ be the set of all functions from ω_1 to $[\omega]^\omega$, and for every $\Psi \in \mathfrak{K}$ let $\mathbb{P}_\Psi$ be the set of all functions g having as domain ω_1 and such that, for every $\alpha \in \omega_1$, $g(\alpha)$ is a chain in $\big([\omega]^\omega, \supseteq^*\big)$ with the following properties:*

a) $\Psi(\alpha) \in g(\alpha)$;

b) $\mathrm{card}\big(g(\alpha)\big) \leq \omega_1$.

Put $\mathbb{P} = \Pi_{\Psi \in \mathfrak{K}} \mathbb{P}_\Psi$, *and for* $p = \{p_\Psi\}_{\Psi \in \mathfrak{K}}$ *and* $q = \{q_\Psi\}_{\Psi \in \mathfrak{K}}$ *let:*

$$p \geq q \Longleftrightarrow \forall \Psi \in \mathfrak{K} \colon \forall \alpha \in \omega_1 \colon p_\Psi(\alpha) \subseteq q_\Psi(\alpha).$$

Then, if G is $\mathbb{P}$-generic over M, we have $(\omega_1)^{M[G]} = (\omega_1)^M$ and in $M[G]$ it holds that for every $\Psi \colon \omega_1 \to [\omega]^\omega$ there is a function $\mathcal{U}$ with domain ω_1 such that:

$$\forall \alpha \in \omega_1 \colon \mathcal{U}(\alpha) \text{ is a non-principal ultrafilter on } \omega \text{ with } \Psi(\alpha) \in \mathcal{U}(\alpha)$$

and

$$\forall S \subseteq \omega_1 \colon \exists \Phi \colon \Bigg(\Phi \text{ is a function } \wedge \ \mathrm{dom}\, \Phi = \omega_1 \\ \wedge \ \big(\forall \alpha \in \omega_1 \colon \Phi(\alpha) \in \mathcal{U}(\alpha)\big) \ \wedge \ \Bigg(\bigcup_{\alpha \in S} \Phi(\alpha)\Bigg) \cap \Bigg(\bigcup_{\alpha \in \omega_1 \setminus S} \Phi(\alpha)\Bigg) = \emptyset \Bigg).$$

Proof. Let us first argue in M. We notice the following two facts.

1) Any two elements $p = \{p_\Psi\}_{\Psi \in \mathfrak{K}}$ and $q = \{q_\Psi\}_{\Psi \in \mathfrak{K}}$ of $\mathbb{P}$ are compatible if and only if $p_\Psi(\alpha) \cup q_\Psi(\alpha)$ is a chain in $\big([\omega]^\omega, \supseteq^*\big)$ for every $\Psi \in \mathfrak{K}$ and $\alpha \in \omega_1$.
2) $\mathbb{P}$ is ω_2-closed. Indeed, if $p^\beta = \{p^\beta_\Psi\}_{\Psi \in \mathfrak{K}}$ is an element of $\mathbb{P}$ for every $\beta \in \omega_1$, and $p^\beta \geq p^\gamma$ for $\beta < \gamma$, then let $p = \{p_\Psi\}_{\Psi \in \mathfrak{K}}$ be defined by $p_\Psi(\alpha) = \bigcup_{\beta \in \omega_1} p^\beta_\Psi(\alpha)$ for every $\Psi \in \mathfrak{K}$ and $\alpha \in \omega_1$. Clearly, $\Psi(\alpha) \in p_\Psi(\alpha)$ for $\Psi \in \mathfrak{K}$ and $\alpha \in \omega_1$; since an increasing union of chains is a chain, and a union of $\leq \omega_1$ many sets of cardinality $\leq \omega_1$ has still cardinality $\leq \omega_1$, we have that $p \in \mathbb{P}$, and of course $p \leq p^\beta$ for every $\beta \in \omega_1$.

Now we leave M and carry out some considerations in $\mathbb{V}$. Let G be $\mathbb{P}$-generic over M: From fact 2) it follows, by a well-known general result (see for example [Ku1, Corollary 6.15]), that

$$(\omega_1)^{M[G]} = (\omega_1)^M.$$

Moreover, every subset of $(\omega_1)^{M[G]} = (\omega_1)^M$ which is in $M[G]$ is also in M (indeed, we may identify subsets of $(\omega_1)^M$ with functions from $(\omega_1)^M$ to 2, and apply [Ku1, Theorem 6.14]). Finally, we also have that every function from $(\omega_1)^{M[G]} = (\omega_1)^M$ to $[\omega]^\omega$, which is in $M[G]$, is in M, as the same argument shows that no new subsets of ω are added by $\mathbb{P}$.

The next arguments take place in $M[G]$. Suppose Ψ be any function from ω_1 to $[\omega]^\omega$: Then, as we have already seen, $\Psi \in \mathfrak{K}$. We may define a function $\mathcal{F}$ by: $\mathcal{F}(\alpha) = \bigcup_{p \in G} p_\Psi(\alpha)$, for every $\alpha \in \omega_1$ (where, for every $p \in \mathbb{P}$ and $\Psi' \in \mathfrak{K}$, $p_{\Psi'}$ denotes the component of p with respect to Ψ'). Since any two elements in G are compatible, it is easily seen that every $\mathcal{F}(\alpha)$ is a chain in $\big([\omega]^\omega, \supseteq^*\big)$. In particular, the intersection of any finite number of elements in $\mathcal{F}(\alpha)$ is infinite, so that $\mathcal{F}(\alpha)$ is a filter base which may be extended to some $\mathcal{U}(\alpha)$, a non-principal ultrafilter on ω. It is clear that $\Psi(\alpha) \in \mathcal{F}(\alpha) \subseteq \mathcal{U}(\alpha)$ for every $\alpha \in \omega_1$ (as $\Psi(\alpha) \in p_\Psi(\alpha)$ for every $p \in \mathbb{P}$).

Now, if we are given a subset S of ${\omega_1}^{M[G]} = {\omega_1}^M$ which belongs to $M[G]$, we also have that $S \in M$. Thus we may work in M and define:

$$D_S = \big\{p \in \mathbb{P} \,\big|\, \exists \Phi\colon \big(\Phi \text{ is a function } \wedge \text{ dom } \Phi = \omega_1 \wedge \\ \forall \alpha \in \omega_1\colon \Phi(\alpha) \in p_\Psi(\alpha) \ \wedge\ \forall \alpha \in S\colon \forall \beta \in \omega_1 \setminus S\colon \Phi(\alpha) \cap \Phi(\beta) = \emptyset\big)\big\}. \tag{1}$$

We prove that D_S is dense in $\mathbb{P}$. Indeed, since $MA(\omega_1)$ holds, we have by a well-known result (see, for example, [Fr1, Chapter 1, §11]) that $\mathfrak{p} > \omega_1$, hence also $\mathfrak{t} > \omega_1$ [vD, Theorem 3.1]. Given any $q \in \mathbb{P}$, we have for every $\alpha \in \omega_1$ that, since $q_\Psi(\alpha)$ is a chain in $\big([\omega]^\omega, \supseteq^*\big)$ with $\operatorname{card}\big(q_\Psi(\alpha)\big) \le \omega_1 < \mathfrak{t}$, there is an $A(\alpha) \in [\omega]^\omega$ which is $\subseteq^*$ of every element of $q_\Psi(\alpha)$. Again, thanks to $MA(\omega_1)$, we have by Lemma 6 that there is a function A' such that $A'(\alpha) \in [A(\alpha)]^\omega$ for every $\alpha \in \omega_1$, and $A'(\alpha) \cap A'(\beta) = \emptyset$ for $\alpha \in S$ and $\beta \in \omega_1 \setminus S$. Then each $A'(\alpha)$ is still almost-contained in every element of $q_\Psi(\alpha)$, so that defining $p_\Psi(\alpha) = q_\Psi(\alpha) \cup \{A'(\alpha)\}$ for every $\alpha \in \omega_1$, and $p_{\Psi'}(\alpha) = q_{\Psi'}(\alpha)$ for every $\Psi' \in \mathfrak{K} \setminus \{\Psi\}$ and $\alpha \in \omega_1$, gives a $p \in D_S$ with $p \le q$.

To finish the proof, let us go back to $M[G]$ and consider an arbitrary $S \subseteq \omega_1$: Then there exists a $p \in G \cap D_S$ (observe that in $M[G]$ the very same definition labelled above as (1) gives rise to the very same set D_S obtained in M, because of the properties of $\mathbb{P}$). Let Φ be such that $\Phi(\alpha) \in p_\Psi(\alpha)$ for $\alpha \in \omega_1$ and $\Phi(\alpha) \cap \Phi(\beta) = \emptyset$ for $\alpha \in S$ and $\beta \in \omega_1 \setminus S$. It turns out that $\Phi(\alpha) \in p_\Psi(\alpha) \subseteq \mathcal{F}(\alpha) \subseteq \mathcal{U}(\alpha)$ for every $\alpha \in \omega_1$; thus Φ is the function we were looking for. □

In the statement and proof of the next corollary we will be working within $M[G]$.

Corollary 8. *In the generic extension $M[G]$ of Theorem 7, for every T_2-topological space (X, τ) which is separable, first-countable and of cardinality ω_1, and every compactification Y of $D(\omega_1)$, there are a topology σ on X with $\sigma \supseteq \tau$, and a continuous function f from a subspace of (X, σ) to Y, such that (X, σ) is a normal, hereditarily extremally disconnected, separable space and f cannot be extended to any continuous $\tilde{f}\colon (X, \sigma) \to Y$.*

Proof. Let E'' be a dense countable subset of (X, τ) and $E' = X \setminus E''$. Index E' in a one-to-one way as $\{x_\alpha\}_{\alpha \in \omega_1}$ and E'' as $\{y_n\}_{n \in \omega}$; then, for every $\alpha \in \omega_1$, fix a countable fundamental system of neighborhoods $\mathcal{V}_\alpha = \{V_\alpha^m \mid m \in \omega\}$ for x_α in (X, τ), with $V_\alpha^{m''} \subsetneqq V_\alpha^{m'}$ for $m' < m''$, and let $\mathcal{N}(\alpha) = \big\{\{n \in \omega \mid y_n \in V_\alpha^m\} \,\big|\, m \in \omega\big\}$. Then $\mathcal{N}(\alpha)$ is a countable chain (with respect to inclusion) in $[\omega]^\omega$, and hence there is $\Psi(\alpha) \in [\omega]^\omega$ such that

$$\forall N \in \mathcal{N}(\alpha)\colon \Psi(\alpha) \subseteq^* N. \tag{$\star$}$$

By Theorem 7, we may associate to every $\alpha \in \omega_1$ a non-principal ultrafilter $\mathcal{U}'(\alpha)$ on ω, with $\Psi(\alpha) \in \mathcal{U}'(\alpha)$, in such a way that for every $S \subseteq \omega_1$ there is a selection Φ of the function $\alpha \mapsto \mathcal{U}'(\alpha)$, with $\big(\bigcup_{\alpha \in S} \Phi(\alpha)\big) \cap \big(\bigcup_{\alpha \in \omega_1 \setminus S} \Phi(\alpha)\big) = \emptyset$. Letting, for every $\alpha \in \omega_1$, $\mathcal{U}(x_\alpha) = \big\{\{y_n \in E'' \mid n \in F\} \,\big|\, F \in \mathcal{U}'(\alpha)\big\}$, we have that condition 1) of Proposition 1 is satisfied—and condition 2) is also satisfied because $|E'| > |E''|$. Therefore, the above-mentioned proposition gives us a topology σ on $X = E' \cup E''$

such that (X, σ) is normal, hereditary extremally disconnected, separable (because E'' is still dense in it), and has the required non-extension property. Moreover, since the ultrafilters $\mathcal{U}'(\alpha)$ are non-principal, $(\star)$ implies that $\mathcal{N}(\alpha) \subseteq \mathcal{U}'(\alpha)$—equivalently, $\mathcal{V}_\alpha \subseteq \mathcal{U}(x_\alpha)$—for every $\alpha \in \omega_1$. This is easily seen to imply $\sigma \supseteq \tau$. □

It is worth observing that Theorem 7 cannot be proved in ZFC. Actually, in ZFC it is impossible to show the existence of even a single $\mathcal{U}$, associating to every element of ω_1 a non-principal ultrafilter $\mathcal{U}(\alpha)$ on ω, in such a way that for every $S \subseteq \omega_1$ there is a selection Φ of $\mathcal{U}$ such that $\left(\bigcup_{\alpha\in S} \Phi(\alpha)\right) \cap \left(\bigcup_{\alpha\in\omega_1\setminus S} \Phi(\alpha)\right) = \emptyset$. Actually, if we assume CH then we can prove that such an association $\alpha \mapsto \mathcal{U}(\alpha)$ cannot exist, because of a combinatorial version of the Jones Lemma (see [Du, Exercise 3 of §VII,3]).

On the other hand, $\alpha \mapsto \mathcal{U}(\alpha)$ with the above properties can be easily constructed assuming the existence of a function $A\colon \omega_1 \to [\omega]^\omega$ with the following property: For every subset S of ω_1, one can find a function $\Psi\colon \omega_1 \to [\omega]^{<\omega}$ with $\big(A(\alpha) \setminus \Psi(\alpha)\big) \cap \big(A(\beta) \setminus \Psi(\beta)\big) = \emptyset$ whenever $\alpha \in S$ and $\beta \in \omega_1 \setminus S$. The existence of an uncountable Q-set Z in the reals implies the existence of such a function A. Indeed, assume Z is a subset of the reals of size ω_1 which is a Q-set. It is well-known that the subset $Y = (Z \times \{0\}) \cup D$ of the Niemytzki plane, where D is a countable dense subset of the upper half of the plane, is normal when Z is a Q-set—see [Ta, Example F]. For every $z \in Z$, let C_z be a sequence of points of D converging to $(z, 0)$. Define a topology τ on Y by declaring all points of D isolated, while taking $\{(\{(z,0)\} \cup C_z) \setminus F \mid F \in [D]^{<\omega}\}$ as a base at the point $(z, 0)$. It is easy to see that this topology is stronger than the original topology on Y, and that Y is τ-normal. Take arbitrary bijections $\phi : \omega_1 \to Z$ and $\eta : D \to \omega$ and define $A(\alpha) = \eta(C_{\phi(\alpha)})$ for $\alpha \in \omega_1$. One can easily verify, using τ-normality of Y, that A has the required property.

3. Spaces with strong extension properties.

In this final section, we want to give some examples of non-trivial spaces X such that every continuous function from a subspace of X to a compact (T_2-)space may be continuously extended to the whole of X (in this case, "non-trivial" means mainly "non-discrete"—or, in a stronger sense, "without isolated points"). This will somehow show that the spaces constructed in the previous two sections are not so common "in nature".

We first point out a basic fact which will play a momentous rôle for the next results, because it will allow us to check the extension property only for functions defined on a closed subset of X.

Lemma 9. *If X is a hereditarily normal, hereditarily extremally disconnected space, then every continuous function from a subspace A of X to a compact (T_2-)space Y may be continuously extended to $\overline{A}$.*

Proof. Use Taĭmanov theorem [En, Theorem 3.2.1] and the characterization of hereditarily extremally disconnected spaces mentioned in the proof of Proposition 1 (separated subsets have disjoint closures). □

Now we recall the notion of *structural normality* [CM, Definition 7.8]. A T_1-space X is said to be structurally normal if it is possible to associate to every

$x \in X$ a fundamental system of open neighborhoods $\mathcal{V}_x$ of x, in such a way that

$$\forall x_1, x_2 \in X\colon \forall V_1 \in \mathcal{V}_{x_1}\colon \forall V_2 \in \mathcal{V}_{x_2}\colon \big((x_1 \notin V_2 \ \wedge\ x_2 \notin V_1) \Longrightarrow V_1 \cap V_2 = \emptyset\big). \quad (*)$$

Structural normality is clearly a hereditary property. Also, if X is structurally normal and we associate to every closed subset C of X the collection $\mathcal{W}_C$ of all sets of the form $\bigcup_{x\in C} V_x$ where, for every $x \in C$, $V_x \in \mathcal{V}_x$, then it is easily seen that each $\mathcal{W}_C$ is a fundamental system of open neighborhoods of C, and that

$$\forall C_1, C_2 \text{ closed in } X\colon \forall W_1 \in \mathcal{W}_{C_1}\colon \forall W_2 \in \mathcal{W}_{C_2}\colon$$
$$(C_1 \cap W_2 = C_2 \cap W_1 = \emptyset \Longrightarrow W_1 \cap W_2 = \emptyset).$$

¿From the above property (which is, in fact, an alternative definition of structural normality) it is easily seen that every structurally normal space is both (hereditarily) collectionwise normal and strongly zero-dimensional.

We are going to prove now that for countable structurally normal spaces, a kind of a very strong version of the Tietze-Urysohn extension theorem (where the unit interval as a co-domain is replaced by an arbitrary topological space) holds.

Proposition 10. *If X is a countable, structurally normal space, then every continuous function f from a closed subspace C of X to a space Y may be extended to a continuous function $\tilde{f}\colon X \to Y$.*

Proof. First of all, suppose to have associated to every $x \in X$ a fundamental system of open neighborhoods $\mathcal{V}_x$ of x, in such a way that condition $(*)$ is satisfied. Since the finite case is trivial, let us write X as $\{x_\ell \mid \ell \in \omega\}$, with $\ell \mapsto x_\ell$ one-to-one. Also, we may clearly suppose $C \neq \emptyset$. Let $S = {}^{<\omega}2$. We will define a subset T of S, a function $\lambda\colon T \to \omega$ and a function Ω from T to the collection of open subsets of X, such that the following conditions are satisfied (where, for every $n \in \omega$, T_n stands for the set $\{s \in T \mid \operatorname{dom} s = n\}$):

1) $T_0 = \{\langle\rangle\}$, $x_{\lambda(\langle\rangle)} \in C$ and $\Omega(\langle\rangle) = X$;

2) $\forall n \in \omega\colon T_{n+1} = \{s^\frown\langle 0\rangle \mid s \in T_n\} \cup \{s^\frown\langle 1\rangle \mid s \in T_n \ \wedge\ \Omega(s) \neq \{x_{\lambda(s)}\}\}$;

3) $\forall n \in \omega\colon \forall s \in T_n\colon \forall \iota \in 2\colon \big(s^\frown\langle\iota\rangle \in T_{n+1} \Longrightarrow \Omega(s^\frown\langle\iota\rangle)$ is a clopen neighbourhood of $x_{\lambda(s^\frown\langle\iota\rangle)}$ included in $\Omega(s)\big)$;

4) $\forall n \in \omega\colon \forall s \in T_n\colon \Big(\Omega(s) \neq \{x_{\lambda(s)}\} \Longrightarrow \big(\lambda(s^\frown\langle 0\rangle) = \lambda(s) \ \wedge\ \lambda(s^\frown\langle 1\rangle) = \min\{n \in \omega \mid x_n \in \Omega(s) \setminus \{x_{\lambda(s)}\}\} \ \wedge\ \Omega(s^\frown\langle 1\rangle) \in \mathcal{V}_{x_{\lambda(s^\frown\langle 1\rangle)}} \ \wedge\ \big(x_{\lambda(s^\frown\langle 1\rangle)} \in X \setminus C \Longrightarrow \Omega(s^\frown\langle 1\rangle) \subseteq X \setminus C\big) \ \wedge\ \Omega(s^\frown\langle 0\rangle) = \Omega(s) \setminus \Omega(s^\frown\langle 1\rangle)\big)\Big)$.

Proceeding inductively, it is straightforward to check that a construction of T, λ and Ω with the above properties may be carried out. As consequences of (1)-(4) we have the following facts.

a) $\forall n \in \omega\colon \forall s \in T_n\colon \big(\Omega(s) = \{x_{\lambda(s)}\} \Longrightarrow (\lambda(s^\frown\langle 0\rangle) = \lambda(s) \ \wedge\ \Omega(s^\frown\langle 0\rangle) = \Omega(s))\big)$.

This trivially follows from (2) and (3).

b) For every $n \in \omega$, the family $\{\Omega(s) \mid s \in T_n\}$ is an open partition of X (indexed in a one-to-one way), and $\{\Omega(s) \mid s \in T_{n+1}\}$ is a refinement of $\{\Omega(s) \mid s \in T_n\}$.

The proof is easily obtained by induction on n, using (1)–(4).

c) For every $\ell \in \omega$, there is a (unique) $\varphi\colon \omega \to 2$, called the *path related to* ℓ, such that $\varphi\restriction_n \in T$ and $x_\ell \in \Omega(\varphi\restriction_n)$ for every $n \in \omega$. We also have that $\Omega(\varphi\restriction_{n'}) \subseteq \Omega(\varphi\restriction_n)$ for $n' \geq n$.

This is a consequence of (b), and of the more precise fact (which follows from (3)) that for every $n \in \omega$ and every $s \in T_n$, the family $\{\Omega(s^\frown\langle\iota\rangle) \mid \iota \in 2 \wedge s^\frown\langle\iota\rangle \in T_{n+1}\}$ is an open partition of $\Omega(s)$.

d) If $\ell \in \omega$ and φ is the path related to ℓ, then for every $s \in T$ with $\lambda(s) = \ell$ we have that $s = \varphi\restriction_n$, where $n = \operatorname{dom} s$.

Indeed, since $x_\ell = x_{\lambda(s)} \in \Omega(s)$ by (1) and (3), and since $x_\ell \in \Omega(\varphi\restriction_n)$ by the definition of path, we may apply (b) to conclude that $\Omega(s) = \Omega(\varphi\restriction_n)$ and that $s = \varphi\restriction_n$.

e) The function $\lambda\colon T \to \omega$ is onto.

To prove this fact, we will show by induction on n that:

$$\forall n \in \omega\colon \forall \ell < n\colon \exists s \in T_n\colon \lambda(s) = \ell.$$

For $n = 0$ everything is trivial, because there is no $\ell < n$. Suppose now that the property holds for $n = \bar{n}$, and let us prove it for $n = \bar{n} + 1$. Let $\ell < \bar{n} + 1$: Then if $\ell < \bar{n}$, we have by the inductive hypothesis that $\ell = \lambda(s)$ for some $s \in T_{\bar{n}}$—hence, using (2), (4) and (a), it follows that $s^\frown\langle 0\rangle \in T_{\bar{n}+1}$ and $\lambda(s^\frown\langle 0\rangle) = \lambda(s) = \ell$. Consider now the case $\ell = \bar{n}$: By (b) we have in particular that there must exist $\hat{s} \in T_{\bar{n}}$ such that $x_{\bar{n}} \in \Omega(\hat{s})$. If $\lambda(\hat{s}) = \bar{n}$ then $\lambda(\hat{s}^\frown\langle 0\rangle) = \bar{n}$, too (apply again (2), (4) and (a)), and we are done; otherwise, $\Omega(\hat{s}) \neq \{x_{\lambda(\hat{s})}\}$ (because $\ell \mapsto x_\ell$ is one-to-one), so that $\hat{s}^\frown\langle 1\rangle \in T_{\bar{n}+1}$ and $\lambda(\hat{s}^\frown\langle 1\rangle) = \min\bigl\{\ell \in \omega \bigm| x_\ell \in \Omega(\hat{s}) \setminus \{x_{\lambda(\hat{s})}\}\bigr\}$ (due to (4)). Observe that $\lambda(\hat{s}^\frown\langle 1\rangle)$ cannot be an $\ell < \bar{n}$ (which would entail that $x_\ell \in \Omega(\hat{s}^\frown\langle 1\rangle)$): Indeed, given an $\ell' < \bar{n}$, by the inductive hypothesis it must be $\lambda(s')$ for some $s' \in T_{\bar{n}}$—whence, like before, $\ell' = \lambda(s'^\frown\langle 0\rangle)$. Therefore $x_{\ell'} \in \Omega(s'^\frown\langle 0\rangle)$, which implies by (b) that $x_{\ell'} \notin \Omega(\hat{s}^\frown\langle 1\rangle)$ (as, of course, $\hat{s}^\frown\langle 1\rangle \neq s'^\frown\langle 0\rangle$). Since $x_{\bar{n}} \in \Omega(\hat{s})$ and $x_{\bar{n}} \neq x_{\lambda(\hat{s})}$ (because we have supposed $\bar{n} \neq \lambda(\hat{s})$, and $\ell \mapsto x_\ell$ is one-to-one), we must necessarily have that $\lambda(\hat{s}^\frown\langle 1\rangle) = \bar{n}$, and we are done.

f) Let $\ell \in \omega$ and φ be the path related to ℓ: Then there exists $n \in \omega$ such that $\lambda(\varphi\restriction_n) = \ell$. Moreover, putting $\bar{n} = \min\{n \in \omega \mid \lambda(\varphi\restriction_n) = \ell\}$, we have that $\varphi(n) = 0$ and $\lambda(\varphi\restriction_n) = \ell$ for every $n \geq \bar{n}$; and if $\bar{n} > 0$, then we also have that $\varphi(\bar{n} - 1) = 1$.

Indeed, we know by (e) that there is $s \in T$ such that $\lambda(s) = \ell$, so that by (d): $s = \varphi\restriction_n$, where $n = \operatorname{dom} s$, and hence $\lambda(\varphi\restriction_n) = \lambda(s) = \ell$. If we further put $\bar{n} = \min\{n \in \omega \mid \lambda(\varphi\restriction_n) = \ell\}$, then let us define inductively s_m for $m \geq 1$ by: $s_1 = \varphi\restriction_{\bar{n}}{}^\frown\langle 0\rangle$ and $s_{m+1} = s_m{}^\frown\langle 0\rangle$. By (2), (4) and (a) we may easily prove by induction that $s_m \in T$ and $\lambda(s_m) = \lambda(\varphi\restriction_{\bar{n}}) = \ell$ for every $m \geq 1$. Then we have by (d) that $s_m = \varphi\restriction_{\operatorname{dom} s_m} = \varphi\restriction_{\bar{n}+m}$ for every $m \geq 1$. Therefore, $\lambda(\varphi\restriction_n) = \lambda(s_{n-\bar{n}}) = \ell$ for every $n > \bar{n}$—hence for every $n \geq \bar{n}$, too; moreover, for $n \geq \bar{n}$ we also have that $\varphi(n) = \varphi\restriction_{n+1}(n) = s_{n-\bar{n}+1}(n) = 0$. Finally, if $\bar{n} > 0$ then we cannot have $\varphi(\bar{n} - 1) = 0$ (or, equivalently, $\varphi\restriction_{\bar{n}}(\bar{n} - 1) = 0$), since otherwise we would obtain by (4) and (a) that $\ell = \lambda(\varphi\restriction_{\bar{n}}) = \lambda(\varphi\restriction_{\bar{n}-1}{}^\frown\langle 0\rangle) = \lambda(\varphi\restriction_{\bar{n}-1})$, contradicting the minimality of $\bar{n}$.

Now we can define our extension $\tilde{f}$. For $x \in C$, we put of course $\tilde{f}(x) = f(x)$. Suppose to have $x_{\hat{\ell}} \in X \setminus C$, and let φ be the path related to $\hat{\ell}$. Then, by (f), $\lambda(\varphi\restriction_n)$ is eventually equal to $\hat{\ell}$. Since $x_{\lambda(\varphi\restriction_0)} = x_{\lambda(\langle\rangle)} \in C$, there exists $\hat{n} = \max\bigl\{n \in \omega \bigm| x_{\lambda(\varphi\restriction_n)} \in C\bigr\}$: Then we put $\tilde{f}(x_{\hat{\ell}}) = f\bigl(x_{\lambda(\varphi\restriction_{\hat{n}})}\bigr)$.

To prove the continuity of $\tilde{f}$, suppose first to have an $x_{\hat{\ell}} \in X \setminus C$: Let φ be the path related to $\hat{\ell}$, and put $\hat{n} = \min\{n \in \omega \mid \lambda(\varphi\restriction_n) = \hat{\ell}\}$. Then $\hat{n} > 0$, and we obtain by (f) that $\varphi(\hat{n}-1) = 1$ and $\lambda(\varphi\restriction_n) = \hat{\ell}$ for every $n \geq \hat{n}$. Thus $\varphi\restriction_{\hat{n}} = \varphi\restriction_{\hat{n}-1}{}^\frown\langle 1\rangle$, and we have by (4) that $\Omega(\varphi\restriction_{\hat{n}}) \subseteq X \setminus C$; we claim that $\tilde{f}\big(\Omega(\varphi\restriction_{\hat{n}})\big) = \big\{\tilde{f}(x_{\hat{\ell}})\big\}$—which clearly implies the continuity of $\tilde{f}$ at $x_{\hat{\ell}}$. Indeed, we know that $\tilde{f}(x_{\hat{\ell}}) = f\big(x_{\lambda(\varphi\restriction_{n^*})}\big)$, where $n^* = \max\{n \in \omega \mid x_{\lambda(\varphi\restriction_n)} \in C\}$; since, as we have already observed, $\lambda(\varphi\restriction_n) = \hat{\ell}$ for every $n \geq \hat{n}$, we have that $n^* < \hat{n}$. Consider now an arbitrary $x_\ell \in \Omega(\varphi\restriction_{\hat{n}})$, and let ψ be the path associated to ℓ: Then $x_\ell \in \Omega(\psi\restriction_{\hat{n}})$, which implies by (d) that $\Omega(\psi\restriction_{\hat{n}}) = \Omega(\varphi\restriction_{\hat{n}})$ and $\psi\restriction_{\hat{n}} = \varphi\restriction_{\hat{n}}$—hence also $\psi\restriction_n = \varphi\restriction_n$ for every $n \leq \hat{n}$. Now, for every $n \geq \hat{n}$ we have by (c) that $\Omega(\psi\restriction_n) \subseteq \Omega(\psi\restriction_{\hat{n}})$, hence $x_{\lambda(\psi\restriction_n)} \in \Omega(\psi\restriction_n) \subseteq \Omega(\psi\restriction_{\hat{n}}) \subseteq X \setminus C$; thus $\max\big\{n \in \omega \,\big|\, x_{\lambda(\psi\restriction_n)} \in C\big\} = \max\big\{n < \hat{n} \,\big|\, x_{\lambda(\psi\restriction_n)} \in C\big\} = \max\big\{n < \hat{n} \,\big|\, x_{\lambda(\varphi\restriction_n)} \in C\big\} = n^*$. Therefore, $\tilde{f}(x_\ell) = \tilde{f}\big(x_{\lambda(\psi\restriction_{n^*})}\big) = \tilde{f}\big(x_{\lambda(\varphi\restriction_{n^*})}\big) = \tilde{f}(x_{\hat{\ell}})$.

Suppose now to have any $x_{\hat{\ell}} \in C$, and let W be a neighborhood in Y of $\tilde{f}(x_{\hat{\ell}}) = f(x_{\hat{\ell}})$. Again, consider the path φ related to $\hat{\ell}$, and let

$$\hat{n} = \min\{n \in \omega \mid \lambda(\varphi\restriction_n) = \hat{\ell}\} : \tag{$\bigstar$}$$

In particular, we will have that $\Omega(\varphi\restriction_{\hat{n}})$ is a clopen neighborhood of $x_{\lambda(\varphi\restriction_{\hat{n}})} = x_{\hat{\ell}}$. By continuity of f, we know that there is a neighborhood V of $x_{\hat{\ell}}$ in X, such that $f(V \cap C) \subseteq W$; and we may further suppose that

$$V \in \mathcal{V}_{x_{\hat{\ell}}} \quad \text{and} \quad V \subseteq \Omega(\varphi\restriction_{\hat{n}}).$$

We claim that $\tilde{f}(V) \subseteq W$. By our choice of V, we only have to show that $\tilde{f}(x) \in W$ whenever $x \in V \setminus C$. Indeed, suppose $x_{\ell^*} \in V \setminus C$, let ψ be the path related to ℓ^* and put

$$n^* = \max\{n \in \omega \mid x_{\lambda(\psi\restriction_n)} \in C\} : \tag{II}$$

Putting $\lambda(\psi\restriction_{n^*}) = \ell^\sharp$, we have by our definition of $\tilde{f}$ that

$$\tilde{f}(x_{\ell^*}) = f(x_{\ell^\sharp}).$$

Since $x_{\ell^*} \in V \subseteq \Omega(\varphi\restriction_{\hat{n}})$, and also $x_{\ell^*} \in \Omega(\psi\restriction_{\hat{n}})$ (because ψ is the path related to ℓ^*), we have by (b) that

$$\psi\restriction_{\hat{n}} = \varphi\restriction_{\hat{n}} .$$

This implies that $x_{\lambda(\psi\restriction_{\hat{n}})} = x_{\lambda(\varphi\restriction_{\hat{n}})} = x_{\hat{\ell}} \in C$, and hence (taking (II) into account):

$$n^* \geq \hat{n}.$$

Let η be the path associated to $\ell^\sharp$: Then $\lambda(\psi\restriction_{n^*}) = \ell^\sharp$ implies by (d) that $\psi\restriction_{n^*} = \eta\restriction_{n^*}$—hence also

$$\psi\restriction_n = \eta\restriction_n \quad \text{for} \quad n \leq n^*.$$

Thus, putting $n^\sharp = \min\{n \in \omega \mid \lambda(\psi\restriction_n) = \ell^\sharp\}$, we will have as well that $n^\sharp = \min\{n \in \omega \mid \lambda(\eta\restriction_n) = \ell^\sharp\}$; then it follows from (f) that $\lambda(\eta\restriction_n) = \ell^\sharp$ for every $n \geq n^\sharp$. Of course, we will have that $n^\sharp \leq n^*$.

Now, if $\ell^\sharp = \hat{\ell}$, then $\tilde{f}(x_{\ell^*}) = f(x_{\ell^\sharp}) = f(x_{\hat{\ell}}) \in W$, and we are done; thus, we may suppose $\ell^\sharp \neq \hat{\ell}$. Notice that this implies:

$$\hat{n} < n^\sharp \tag{Γ}$$

and

$$x_{\hat{\ell}} \notin \Omega(\eta\restriction_{n^\sharp}). \tag{Λ}$$

Indeed, to prove (Γ), suppose by contradiction $\hat{n} \geq n^\sharp$. Then, as we have already observed, it follows that $\lambda(\eta\restriction_{\hat{n}}) = \ell^\sharp$. On the other hand, since $\hat{n} \leq n^*$, we have that $\psi\restriction_{\hat{n}} = \eta\restriction_{\hat{n}}$, so that $\lambda(\psi\restriction_{\hat{n}}) = \ell^\sharp$. But this is impossible because $\psi\restriction_{\hat{n}} = \varphi\restriction_{\hat{n}}$ and hence $\lambda(\psi\restriction_{\hat{n}}) = \lambda(\varphi\restriction_{\hat{n}}) = \hat{\ell}$ by ($\bigstar$).

Suppose now that (Λ) fails, i.e. $x_{\hat{\ell}} \in \Omega(\eta\restriction_{n^\sharp})$: Since $x_{\hat{\ell}} \in \Omega(\varphi\restriction_{n^\sharp})$ (because φ is the path of $\hat{\ell}$), we have by (b) that $\varphi\restriction_{n^\sharp} = \eta\restriction_{n^\sharp}$—hence also $\varphi\restriction_{n^\sharp} = \psi\restriction_{n^\sharp}$. But the definition of $\hat{n}$ (see ($\bigstar$)) implies by (f) that $\lambda(\varphi\restriction_n) = \hat{\ell}$ for every $n \geq \hat{n}$; since $n^\sharp > \hat{n}$ by (Γ), we obtain that $\lambda(\psi\restriction_{n^\sharp}) = \lambda(\varphi\restriction_{n^\sharp}) = \hat{\ell}$, while $\lambda(\psi\restriction_{n^\sharp}) = \ell^\sharp$ by definition of $n^\sharp$. A contradiction.

Now, since $n^\sharp > \hat{n}$ implies in particular that $n^\sharp > 0$, and $n^\sharp = \min\{n \in \omega \mid \lambda(\eta\restriction_n) = \ell^\sharp\}$, we have by (f) that $\eta(n^\sharp - 1) = 1$—hence also $\psi(n^\sharp - 1) = 1$, because $n^\sharp - 1 < n^\sharp \leq n^*$ and $\eta\restriction_{n^*} = \psi\restriction_{n^*}$. By (4), we obtain that $\Omega(\psi\restriction_{n^\sharp}) = \Omega\left(\psi\restriction_{n^\sharp-1}{}^\frown\langle 1\rangle\right) \in \mathcal{V}_{x_{\lambda(\psi\restriction_{n^\sharp})}} = \mathcal{V}_{x_{\ell^\sharp}}$. Since ψ is the path of ℓ^*, x_{ℓ^*} must belong to $\Omega(\psi\restriction_{n^\sharp})$; but $x_{\ell^*} \in V$, too, by its initial choice. It follows that $V \cap \Omega(\psi\restriction_{n^\sharp}) \neq \emptyset$, which implies (as $V \in \mathcal{V}_{x_{\hat{\ell}}}$ and $\Omega(\psi\restriction_{n^\sharp}) \in \mathcal{V}_{x_{\ell^\sharp}}$) that either $x_{\hat{\ell}} \in \Omega(\psi\restriction_{n^\sharp})$ or $x_{\ell^\sharp} \in V$—cf. property $(*)$ in the definition of structural normality. Since the first relation is in contrast with (Λ), we have that $x_{\ell^\sharp} \in V$—hence also $x_{\ell^\sharp} \in V \cap C$, by our definitions of $\ell^\sharp$ and n^*. Therefore, $\tilde{f}(x_{\ell^*}) = f(x_{\ell^\sharp}) \in f(V \cap C) \subseteq W$. $\square$

Corollary 11. *If X is a structurally normal, hereditarily extremally disconnected, hereditarily separable space, then every continuous function f from a subspace of X to a compact (T_2-)space Y may be extended to a continuous $\tilde{f}\colon X \to Y$.*

Proof. Let $f\colon M \to Y$, with $M \subseteq X$, be continuous: Since $\overline{M}$ is hereditarily normal and hereditarily extremally disconnected, by Lemma 9 we may extend f to a continuous $f^*\colon \overline{M} \to Y$. Fix a countable dense subset C of $\overline{M}$ and a countable dense subset D of $X \setminus \overline{M}$: then C is closed in $X^\sharp = C \cup D$ and $X^\sharp$ is a countable, structurally normal space. By Proposition 10, there must exist a continuous extension $f^\sharp\colon X^\sharp \to Y$ of $f^*\restriction_C$; and since $X^\sharp$ is obviously dense in X, we may apply Lemma 9 again to get a continuous $\tilde{f}\colon X \to Y$ which extends $f^\sharp$. Thus, it only remains to show that $\tilde{f}$ is an extension of f: This will immediately follow if we can prove that $\tilde{f}\restriction_{\overline{M}} = f^*$. Indeed, since $\tilde{f}\restriction_{\overline{M}}$ and f^* are both continuous, and they coincide on the dense subset C of $\overline{M}$, they must coincide also on the whole of $\overline{M}$. $\square$

We will now give some examples of spaces satisfying the hypotheses on X, in the statement of the above result. Besides countable discrete spaces, the simplest non-trivial example seems to be a subspace of $\beta\omega$, given by ω plus a point p of ω^*. This space is trivially seen to be structurally normal and (hereditarily) extremally disconnected.

In [CM], after Definition 7.8, there is an example of a countable, structurally normal, hereditarily extremally disconnected space X with no isolated points. The construction of this space is performed in the following way. One takes as X the set ${}^{<\omega}\omega$; then one fixes a non-principal ultrafilter $\mathcal{U}$ on ω and defines a subset A of X to be open if and only if, for every $s \in A$, the set $\{n \in \omega \mid s^\frown\langle n\rangle \in A\}$ belongs to $\mathcal{U}$. Since the topological properties of X are just stated in [CM], without any real proof, we will provide it here for what concerns structural normality and hereditary extremal disconnectedness.

It is easily seen that set-theoretic inclusion, when restricted to X, gives it a tree structure. We will say that a subset M of X is *tree-convex* if for any two $s', s'' \in X$ with $s' \subseteq s''$, we have that

$$\forall m \in \omega\colon \big(\operatorname{dom} s' \le m \le \operatorname{dom} s'' \implies s''\restriction_m \in M\big).$$

For every $s \in X$, we will put

$$\mathcal{V}_s = \{A \subseteq X \mid A \text{ is open and tree-convex} \quad \wedge \quad s = \min A \text{ with respect to set-theoretic inclusion}\}.$$

Proposition 12. *For every $s \in X$, $\mathcal{V}_s$ is a fundamental system of open neighborhoods for s, and the association $s \mapsto \mathcal{V}_s$ satisfies property $(*)$ in the definition of structural normality.*

Proof. Let $\hat{s} \in X$, and Ω be an open set containing $\hat{s}$. By our definition of the topology on X, and applying the axiom of choice, we have that there must exist a function $U\colon \Omega \to \mathcal{U}$ such that:

$$\forall s \in \Omega\colon \{s^\frown\langle n\rangle \mid n \in U(s)\} \subseteq \Omega.$$

Then define by induction the sets $A_m \subseteq \Omega$ by:

$$A_0 = \{\hat{s}\}; \quad A_{m+1} = \{s^\frown\langle n\rangle \mid s \in A_m \quad \wedge \quad n \in U(s)\}.$$

It is easily seen that $A = \bigcup_{m\in\omega} A_m$ is open, is included in Ω, and that $\hat{s}$ is the minimum of A with respect to set-theoretic inclusion. To prove tree-convexity, let $s', s'' \in A$ with $s' \subseteq s''$. Put $F = \{n \in \omega \mid \operatorname{dom} s' \le n \le \operatorname{dom} s''\}$ and suppose towards a contradiction that $F \setminus \{n \le \operatorname{dom} s'' \mid s''\restriction_n \in A\} \ne \emptyset$: Then we may consider the maximum $\bar{n}$ of this set. Since $\bar{n} < \operatorname{dom} s''$, we have that $\bar{n}+1 \in F$ and hence $s''\restriction_{\bar{n}+1} \in A$. Also, $s''\restriction_{\bar{n}+1}$ cannot coincide with $\hat{s}$ (because $\hat{s}$ has minimum domain in A, while $\operatorname{dom}(s''\restriction_{\bar{n}+1}) = \bar{n}+1 > \bar{n} \ge \operatorname{dom} s'$), hence $s''\restriction_{\bar{n}+1} \in A_{\bar{m}}$ for some $\bar{m} > 0$. Thus, by the definition of the sets A_m, we have that $s''\restriction_{\bar{n}} = (s''\restriction_{\bar{n}+1})\restriction_{\bar{n}} \in A_{\bar{m}-1} \subseteq A$, contradicting the definition of $\bar{n}$.

Now we prove that $s \mapsto \mathcal{V}_s$ witnesses the structural normality of X. Suppose $s_0, s_1 \in X$, $A_0 \in \mathcal{V}_{s_0}$ and $A_1 \in \mathcal{V}_{s_1}$: If $A_0 \cap A_1 \ne \emptyset$, let $s^* \in A_0 \cap A_1$. Then for every $\iota \in 2$, since $s_\iota = \min A_\iota$, we have that $s_\iota \subseteq s^*$, i.e. $s_\iota = s^*\restriction_{n_\iota}$ where $n_\iota = \operatorname{dom} s_\iota$. Let $\hat{\iota} \in 2$ be such that $n_{\hat{\iota}} = \min\{n_0, n_1\}$: We have that $s_{1-\hat{\iota}}\restriction_{n_{\hat{\iota}}} = (s^*\restriction_{n_{1-\hat{\iota}}})\restriction_{n_{\hat{\iota}}} = s^*\restriction_{n_{\hat{\iota}}} = s_{\hat{\iota}}$, whence $s_{\hat{\iota}} \subseteq s_{1-\hat{\iota}}$. Since $s_{\hat{\iota}}, s^* \in A_{\hat{\iota}}$ and $s_{\hat{\iota}} \subseteq s_{1-\hat{\iota}} \subseteq s^*$, $s_{1-\hat{\iota}}$ must belong to $A_{\hat{\iota}}$ because of tree-convexity. Therefore, we have proved that $A_0 \cap A_1 \ne \emptyset \implies (s_1 \in A_0 \quad \vee \quad s_0 \in A_1)$, which means that condition $(*)$ in the definition of structural normality is satisfied. □

Since we have chosen the ultrafilter $\mathcal{U}$ to be non-principal, X is clearly T_1—hence by the above proposition it is structurally normal. Now we are going to prove hereditary extremal disconnectedness of X. Since structural normality implies hereditary normality, by the second remark after Proposition 1 it will be sufficient to show extremal disconnectedness.

Lemma 13. *Let M be any subset of X, and $\hat{s} \in \overline{M} \setminus M$. Then $\{n \in \omega \mid \hat{s}^\frown\langle n\rangle \in \overline{M}\} \in \mathcal{U}$.*

Proof. Towards a contradiction, suppose $F = \{n \in \omega \mid \hat{s}^\frown\langle n\rangle \in \overline{M}\} \notin \mathcal{U}$: Then $\omega \setminus F \in \mathcal{U}$. Let $A = (X \setminus \overline{M}) \cup \{\hat{s}\}$: We claim that A is open, and this will lead to a contradiction as $\hat{s}$ cannot have any neighborhood disjoint from M.

To prove that A is open, we have to show that

$$\forall s \in A\colon \{n \in \omega \mid s^\frown\langle n\rangle \in A\} \in \mathcal{U}. \tag{$\sharp$}$$

Actually, given any $s \in A$, we have that $\{n \in \omega \mid s^\frown\langle n\rangle \in X \setminus \overline{M}\} \in \mathcal{U}$—which clearly implies ($\sharp$). Indeed, if $s \neq \hat{s}$, then s belongs to $X \setminus \overline{M}$, which is open; and if $s = \hat{s}$, then the above relation follows from the fact that $\omega \setminus F \in \mathcal{U}$. □

Proposition 14. *X is extremally disconnected.*

Proof. Let A be open in X and $\hat{s} \in \overline{A}$. If $\hat{s} \in A$, then $\{n \in \omega \mid \hat{s}^\frown\langle n\rangle \in A\} \in \mathcal{U}$—hence $\{n \in \omega \mid \hat{s}^\frown\langle n\rangle \in \overline{A}\} \in \mathcal{U}$, too. And if $\hat{s} \in \overline{A} \setminus A$, then we can clearly apply the previous lemma to get the same result. □

4. Open questions.

We do not know whether structural normality can be weakened to hereditary normality in Corollary 11.

Question 15. *Assume that X is a hereditarily normal, hereditarily separable, hereditarily extremally disconnected space, A is a subspace of X, K is a compact space and $f : A \to K$ is a continuous function. Can then f be extended to a continuous function $\tilde{f} : X \to K$?*

Even the following concrete case of the previous question seems interesting.

Question 16. *Assume that X is a countable, hereditarily extremally disconnected, regular space, A a subspace of X, K a compact space and $f : A \to K$ a continuous function. Can then f be extended to a continuous function $\tilde{f} : X \to K$?*

We do not know whether the requirement of structural normality of X can be omitted in Proposition 10.

Question 17. *Assume that X is a countable regular space, A is a closed subspace of X, Y is an arbitrary space and $f : A \to Y$ is a continuous function. Can then f be extended to a continuous function $\tilde{f} : X \to Y$?*

Remark. Since a countable regular space is (hereditarily) normal, the space X in both Questions 16 and 17 is (hereditarily) normal. Of course, if we had not assumed normality of X, then the answer to both questions would have been negative, in the first case by [CM, Proposition 7.4 or Theorem 7.5], and in the second case because if C_0, C_1 are closed disjoint subsets of X which cannot be separated by open sets, then $f\colon C_0 \cup C_1 \to \{0, 1\}$ defined by $f(x) = \iota$ for $x \in C_\iota$ cannot be extended to any continuous function $\tilde{f}\colon X \to \{0, 1\}$.

An example of a countable, hereditarily extremally disconnected Hausdorff space which is not (semi)regular will be given in the end of this manuscript.

Question 18. *In ZFC, does there exist a separable, hereditarily normal, hereditarily extremally disconnected space X such that some continuous function f from a subset of X to a compact space Y cannot be continuously extended to a function $\tilde{f} : X \to Y$?*

Problem 19. *Let Y be a space with the following property: For every hereditarily normal, hereditarily extremally disconnected space X, each subspace A of X and any continuous function $f : A \to Y$, there exists a continuous function $\tilde{f} : X \to Y$ extending f. Must then Y be metrizable?*

By Proposition 7.7 from [CM], Y must be compact. A positive answer to Problem 19 combined with Theorem 7.5 from [CM] would provide a nice characterization of compact metric spaces in terms of extensions of continuous functions.

Recall that a space X is said to be *semiregular* if each of its points has a fundamental system of neighbourhoods consisting of *open domains*, i.e. sets which coincide with the interior of their closure.

We finish this paper with the example promised in the remark after Question 17.

Example 20. Let $\{S_n \mid n \in \omega\}$ be a faithfully indexed family of pairwise disjoint infinite subsets of ω, and let each $\mathcal{U}_n$ be a non-principal ultrafilter on ω such that $S_n \in \mathcal{U}_n$. Fix another non-principal ultrafilter $\mathcal{U}$ on ω and observe that

$$\mathcal{U}_\infty = \big\{S \subseteq \omega \;\big|\; \{n \in \omega \mid S \in \mathcal{U}_n\} \in \mathcal{U}\big\}$$

is a non-principal ultrafilter on ω. Define $X = \omega \cup \big\{x_\ell \,\big|\, \ell \in \omega \cup \{\infty\}\big\}$, where $\ell \mapsto x_\ell$ is one-to-one map and $\omega \cap \big\{x_\ell \,\big|\, \ell \in \omega \cup \{\infty\}\big\} = \emptyset$. Endow X with the topology making every point $n \in \omega$ isolated, while each x_ℓ has $\big\{\{x_\ell\} \cup M \,\big|\, M \in \mathcal{U}_\ell\big\}$ as its fundamental system of (open) neighbourhoods. Then X *is a (countable) hereditarily extremally disconnected Hausdorff space, which is not semiregular.*

Proof. Let us check relevant properties of X.

X is Hausdorff. If $\ell', \ell'' \in \omega$ are distinct, then $\{x_{\ell'}\} \cup S_{\ell'}$ and $\{x_{\ell''}\} \cup S_{\ell''}$ are disjoint neighbourhoods of $x_{\ell'}$ and $x_{\ell''}$, respectively. To separate an x_ℓ with $\ell \in \omega$ from x_∞, simply observe that $\omega \setminus S_\ell$ belongs to $\mathcal{U}_\infty$ because

$$\{\ell' \in \omega \mid \omega \setminus S_\ell \in \mathcal{U}_{\ell'}\} = \omega \setminus \{\ell\} \in \mathcal{U}$$

(recall that we have chosen $\mathcal{U}$ to be non-principal).

X is hereditarily extremally disconnected. Let A, B be subsets of X with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. By contradiction, suppose that $y \in \overline{A} \cap \overline{B}$. Then $y \notin A$ and $y \notin B$, so that $y = x_\ell$ for some $\ell \in \omega \cup \{\infty\}$ and $y \in \overline{(A \cap \omega)} \cap \overline{(B \cap \omega)}$ (this is an obvious consequence of the way we have defined the topology on X). But $x_\ell \in \overline{A \cap \omega}$ implies that $A \cap \omega \in \mathcal{U}_\ell$ (otherwise $\{x_\ell\} \cup (\omega \setminus A)$ would be a neighbourhood of x_ℓ disjoint from $A \cap \omega$), and analogously $x_\ell \in \overline{B \cap \omega}$ implies that $B \cap \omega \in \mathcal{U}_\ell$. Since A and B are disjoint, this is a contradiction.

X is not semiregular. Since a semiregular hereditarily extremally disconnected space is regular (see, for example, [En, Exercise 6.3.18]), it suffices to show that X is not regular. The set $C = \{x_\ell \mid \ell \in \omega\}$ is clearly closed in X. We prove that every neighbourhood of x_∞ contains some element of C in its closure. Indeed, let $\{x_\infty\} \cup M$ be an arbitrary (basic) neighbourhood of x_∞ with $M \in \mathcal{U}_\infty$. Then $L = \{\ell \in \omega \mid M \in \mathcal{U}_\ell\} \in \mathcal{U}$. Hence $L \neq \emptyset$, and $x_\ell \in \overline{M} \subseteq \overline{\{x_\infty\} \cup M}$ for every $\ell \in L$. $\square$

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