BENJAMIN-ONO 方程式の解の存在と一意性について

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In this talk, we consider the existence and the uniqueness of solutions to the Benjamin-Ono equation,

$$(1) \qquad \begin{cases} \partial_t u + \mathrm{H}\partial_x^2 u + \frac{1}{2}\partial_x(u^2) = 0, & \text{in } \mathbb{R}\times\mathbb{R}, \\ u(0,x) = \phi(x), & \text{in } \mathbb{R}, \end{cases}$$

where H is the Hilbert transform which is defined by

$$\mathrm{H}f = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{x-y}dy = \mathcal{F}^{-1}(-i\,sgn(\xi))\mathcal{F}f,$$

and $\mathcal{F}$ denotes the Fourier transform with respect to x.

Definition 1. Let s_1, s_2, b_1 and b_2 be real numbers. We define a function space $X^{s_1,s_2}_{b_1,b_2}$ as follows;

$$(2) \quad X^{s_1,s_2}_{b_1,b_2} =$$
$$\left\{ f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X^{s_1,s_2}_{b_1,b_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\langle\tau+\xi^2\rangle^{b_1}\langle\tau-\xi^2\rangle^{b_2}\hat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} < +\infty \right\}.$$

Here $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ and $\hat{f}(\tau,\xi)$ is the Fourier transform of $f(t,x)$ with respect to space and time variables.

We shall find a solution to the associate integral equation of (1),

$$(3) \qquad u(t) = U(t)\phi + \int_0^t U(t-s)\partial_x u(s)^2 ds,$$

instead of the intial value problem (1) directly. Here $U(t)\phi = \exp(-t\mathrm{H}\partial_x^2)\phi$ $= \mathcal{F}^{-1}\exp(-it\xi|\xi|)\mathcal{F}\phi$. Let ψ be a function in $C_0^\infty(\mathbb{R})$ with $0 \le \psi \le 1$, $\psi(t) = 1$ for $|t| \le 1$ and $\psi(t) = 0$ for $|t| \ge 2$. We consider the following integral equation,

$$(4) \qquad u(t,x) = \psi(t)U(t)\phi + \psi(t)\int_0^t U(t-s)\partial_x u(s)^2 ds.$$

Definition 2. Let s_1 and s_2 be real numbers. Function space $H^{s_1,s_2}(\mathbb{R}) = H^{s_1,s_2}$ is defined by

(5)
$$H^{s_1,s_2}(\mathbb{R}) = \{g(x) \in \mathcal{S}'(\mathbb{R}); \|g\|_{H^{s_1,s_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\hat{g}(\xi)\|_{L^2} < +\infty\}.$$

Our main theorem is the following.

Theorem 1. *Suppose that* $\delta > 0$, $\phi \in H^{1+\delta,-1/2}(\mathbb{R})$ *and* $\|\phi\|_{H^{0,-1/2}} \ll 1$. *Then there exists a unique solution* $u(t,x)$ *to the integral equation* (4) *in* $X^{\delta,-1/2}_{1/2,1/2}$. *Moreover, we have*

$$\|u_1(t,x) - u_2(t,x)\|_{X^{\delta,-1/2}_{1/2,1/2}} \leq C\|\phi_1 - \phi_2\|_{H^{1+\delta,-1/2}},$$

where u_j *is a solution to the equation* (4) *with initial data* ϕ_j *for* $j = 1, 2$.

J. Bourgain[2] has shown L^2 local wellposedness for the Korteweg-de Vries equation. Kenig-Ponce-Vega[5] has extended this result to H^s local wellposedness with $s > -3/4$. For the Benjamin-Ono equation, L. Abdelouhab, J. L. Bona, M. Felland and J. C. Saut[1] and Iorio Jr.[3] has shown global wellposedness for $s > 3/2$. Ponce[8] has shown global wellposedness for $s = 3/2$. Recently Koch and Tvetkov[7] has shown local wellposedness for $s > 5/4$. Very recently, C. E. Kenig and K. D. Koenig[4] has shown local wellposedness for $s > 9/8$. T. Tao[9] has shown local and global wellposedness for $s = 1$.

These results above use a priori estimate and compactness argument. Our method used in this talk is the iteration method in some Sobolev spaces mixed between homogenious and inhomogenious Sobolev spaces which is defined in the definition 2. In this direction, N. Kita and J. Segata[6] has recently shown the wellposedness of solutions for the weighted Sobolev space by the iteration method, which consists of functions satisfying that $\phi \in H^s$ with $s > 1$ and $\langle x\rangle^\alpha \phi \in H^{s_1}$ with $s_1 + \alpha < s$, $1/2 < s_1$ and $1/2 < \alpha < 1$.

We prove this theorem by combining the following lemmas.

Lemma 1. *For* $\phi \in \mathcal{S}(\mathbb{R})$, *we have*

$$\|\psi(t)U(t)\phi\|_{X^{s_1,s_2}_{b,b}} \leq C\|\phi\|_{H^{s_1+2b,s_2}}.$$

Lemma 2. *For* $f(t,x) \in \mathcal{S}(\mathbb{R}^2)$, *we have*

$$\|\psi(t)\int_0^t U(t-s)f(s,x)ds\|_{X^{s_1,s_2}_{1/2,1/2}} \leq C\left(\|f\|_{X^{s_1,s_2}_{-1/2,1/2}} + \|f\|_{X^{s_1,s_2}_{1/2,-1/2}} + \|f\|_{Y^{s_1+1,s_2}}\right), \tag{6}$$

where

$$\|f\|_{Y^{s_1+1,s_2}} = \left(\int_{-\infty}^{\infty} \langle\xi\rangle^{2s_1+2}|\xi|^{2s_1}\left(\int_{-\infty}^{\infty}\frac{|\hat{f}(\tau,\xi)|}{\langle\tau+\xi|\xi|\rangle}d\tau\right)^2 d\xi\right)^{1/2}. \tag{7}$$

Lemma 3. *For $\delta > 0$, the following inequalities hold for $f, g \in \mathcal{S}(\mathbb{R}^2)$:*

$$\|\partial_x(fg)\|_{X^{\delta,-1/2}_{-1/2,1/2}} \leq C\|f\|_{X^{\delta,-1/2}_{/2,1/2}}\|g\|_{X^{\delta,-1/2}_{/2,1/2}},$$

$$\|\partial_x(fg)\|_{X^{\delta,-1/2}_{1/2,-1/2}} \leq C\|f\|_{X^{\delta,-1/2}_{1/2,1/2}}\|g\|_{X^{\delta,-1/2}_{1/2,1/2}},$$

$$\|\partial_x(fg)\|_{Y^{1+\delta,-1/2}} \leq C\|f\|_{X^{\delta,-1/2}_{1/2,1/2}}\|g\|_{X^{\delta,-1/2}_{1/2,1/2}}.$$

Remark 1. $H^{1+\delta,-1/2} \approx H^{1/2+\delta}$ *in high frequency region.*

Remark 2.

$$X^{\delta,-1/2}_{1/2,1/2} \subset C(\mathbb{R}; H^{1+\delta-\epsilon,-1/2})$$

for $\epsilon > 0$.

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