ON A THEOREM OF MISLIN ON COHOMOLOGY ISOMORPHISM AND CONTROL OF FUSION

Introduction

Let $kG$ be the group algebra of a finite group $G$ over an algebraically closed field $k$ of characteristic $p > 0$. In 1990 [9] G.Mislin proved the following remarkable theorem.

**Theorem (Mislin).** Let $H$ be a subgroup of $G$. Then the restriction map in mod-$p$ cohomology $\text{res}_H^G : H^\ast(G, k) \to H^\ast(H, k)$ is an isomorphism if and only if $H$ controls strong $p$-fusion in $G$.

"If" part in the theorem has long been known to be true. For "Only if" part Mislin’s proof uses deep results from algebraic topology. In 2001 [11] V.P.Snaith gave an alternating proof of Mislin’s theorem which uses also topological results. In [10] G.R.Robinson remarked that Mislin’s theorem can be obtained if one could prove the non-vanishing of cohomology of certain types of trivial source $kG$-modules.

Isomorphism classes of indecomposable trivial source $kG$-modules are parametrized as follows. Let $P$ be a $p$-subgroup of $G$ and $S$ be a simple $kN_G(P)$-module. Let $M_{P,S}^{N_G(P)}$ be a projective cover of $S$ as $kN_G(P)/P$-module. Inflating $M_{P,S}^{N_G(P)}$ to $kN_G(P)$ and taking its Green correspondent, we obtain an indecomposable trivial source module $M_{P,S}^G$ with vertex $P$. And each indecomposable trivial source module is obtained in this way.

P.Symonds in [13] proved the following result from which Mislin’s theorem is obtained following Robinson’s remark.

**Theorem (Symonds).** In the notations above, $H^\ast(G, M_{P,S}^G) \neq 0$ if and only if $C_G(P)$ acts trivially on $S$.

A proof of the above theorem given by P.Symonds needs also topological methods. My aim in this talk is to give an algebraic proof of the theorem of P.Symonds.

A.Hida [8] also obtained an algebraic proof of the above Symonds’ theorem and explained his idea in his talk at this meeting. A very elegant proof !!

In my lecture I first introduced the idea of Robinson to find an algebraic proof of Mislin’s theorem and how his idea relates Symonds’ theorem. This is included in section 1 in this note. And then I discussed the theorem of Symonds. In the lecture I only gave an outline of my proof of the theorem. I shall give my proof in detail in this note.
"Only if" part of the theorem has been essentially proved by Benson, Carlson and Robinson in [5]. In section 2 in this note we shall give a proof of "Only if part" following arguments by them.

For "If" part we first reduce the problem to some $p$-local subgroup. This is done in section 3. Our $p$-local subgroup is a normalizer of some elementary abelian $p$-group. Then we use the idea of Symonds in [12] to find a nonzero cohomology element. There he made use of the Lyndon-Hochschild-Serre spectral sequence and some result on the action of Aut$(E)$ on the cohomology algebra $H^*(E,k)$, where $E$ is an elementary abelian $p$-group. He needed also a result of Duflot [6] on the depth of cohomology algebras of groups with central elementary abelian groups. For these results there has been given algebraic proofs (see for example [2],[4] and [7]) and we believe that our proof of the theorem is an algebraic one.

1. Robinson’s Idea

In this section let $H$ be a subgroup of $G$ and assume that $\text{res}_H^G : H^*(G, k) \rightarrow H^*(H, k)$ is an isomorphism. We first remark the following.

Lemma 1.1. $H$ contains a Sylow $p$-subgroup of $G$.

Proof. Consider an $H$-injective hull of $k_G; 0 \rightarrow k_G \xrightarrow{f} k_H \xrightarrow{\iota} L \rightarrow 0$. We obtain the following long exact sequence

$$- \rightarrow H^n(G, k) \xrightarrow{f^*} H^n(G, k_H \uparrow^G) \rightarrow H^n(G, L) \rightarrow H^{n+1}(G, k) \xrightarrow{f^*} H^{n+1}(G, k_H \uparrow^G) \rightarrow$$

Identify $H^n(G, k_H \uparrow^G)$ with $H^n(H, k_H)$ by Eckmann-Shapiro. Then it follows that the map $f_*$ coincides with the restriction map $\text{res}_H^G$. By our assumption we have $H^n(G, L)$ for $n \geq 0$. By a theorem of Benson-Carlson-Robinson (Theorem 2.4 [5]), $\hat{H}^n(G, L) = 0$ for all $n$, where $\hat{H}^n$ is Tate’s cohomology. In particular, $\text{res}_H^G : \hat{H}^{-1}(G, k_G) \rightarrow \hat{H}^{-1}(H, k_H)$ is an isomorphism. Any non zero element in $\hat{H}^{-1}(G, k)$ represents the almost split sequence terminating at $k_G$ and it is well known that the sequence does not split as a sequence of $kH$-modules if and only if $H$ contains a Sylow $p$-subgroup of $G$. Thus the lemma is proved.

Assume that $H$ contains a Sylow $p$-subgroup of $G$ and $H$ does not control $p$-fusion. Then there exists a $p$-subgroup $P$ of $H$ such that $N_G(P) \supseteq C_G(P)N_H(P)$. Choose $P$ maximal with this property, then $C_G(P) = Z(P) \times O_P(C_G(P))$ and $C_G(P)N_H(P)/P$ is a strongly $p$-embedded subgroup of $N_G(P)/P$. Set $C = C_G(P)N_H(P)$. Then $k_C \uparrow_{N_G(P)}^C = k \oplus M$ for some $kN_G(P)$-module $M$. Each indecomposable summand of $M$ has the form $M_{P, S}^{N_G(P)}$ with $C_G(P) \subset \text{Ker} S$. $(k_H \uparrow^G) \downarrow_{N_G(P)} = k_N(H)^{k_H \uparrow^G} \uparrow_{N_G(P)}^C \oplus U = k_C \uparrow_{N_G(P)}^C \oplus U' = k_G \oplus M \oplus U'$ for some $kN_G(P)$-modules $U, U'$. By a theorem of Burry-Carlson, $k_H \uparrow^G = k_G \oplus M_{P, S}^{N_G(P)} \oplus V$ with $\text{Ker} S \supset C_G(P)$.

Now Symonds’ theorem implies that $H^G(M_{P, S}^{N_G(P)}) \neq 0$ and we can conclude that $H^*(H, k) = H^*(G, k_H \uparrow^G) \supseteq H^*(G, k)$ and the "only if" part of Mislin’s theorem follows.
2. Proof of “Only if” Part

Let $P$ be a $p$-subgroup of $G$ and $S$ be a simple $kN_G(P)$-module. And let $M_{P,S}^G$ be an indecomposable $kG$-module with vertex $P$ and with trivial source described in introduction. In these notations we shall prove the following.

**Theorem 2.1.** $H^*(G, M_{P,S}^G) = 0$ if $C_G(P)$ acts nontrivially on $S$.

We argue following a proof of Proposition 5.3 in [5]. If $C_G(P)$ acts nontrivially on $S$, then there exists a $p'$-element $y \neq 1$ in $C_G(P)$ such that $y$ acts nontrivially on $S$. Thus there exists a one dimensional submodule $M_0$ of $S_{(y) \times P}$ on which $y$ acts nontrivially. Then $M_0 \uparrow^{N_G(P)}$ has a summand isomorphic to $M_{P,S}^{N_G(P)}$ because $M_0 \uparrow^{N_G(P)}$ is a projective $kN_G(P)/P$-module and $\text{Hom}_{kN_G(P)}(M_0 \uparrow^{N_G(P)}, S) \cong \text{Hom}_{k_{(y) \times P}}(M_0, S_{(y) \times P}) \neq 0$. Therefore $M = M_{P,S}^G$ appears in summand of $M_0 \uparrow^G$ and $H^*(G, M) \leq H^*(G, M_0 \uparrow^G)$. Now the result follows by Lemma 5.1 in [5].

3. Proof of ”If” Part

Let $H$ be a subgroup of $G$ and $P$ be a $p$-subgroup of $H$. Then the module $M_{P,k}^H$ where $k = k_{N_H(P)}$ is the trivial $kN_H(P)$-module is called a Scott module of $H$ with vertex $P$ and we shall denote it by $SC_P^H$. It is well known that $SC_P^H$ is a unique trivial source module of $H$ with vertex $P$ which contains $k_H$.

Throughout this section let $M = M_{P,S}^G$ where $P$ is a $p$-subgroup of $G$ and $S$ is a simple $kN_G(P)$-module on which $C_G(P)$ acts trivially. Notice that the condition that $C_G(P)$ acts trivially on $S$ is equivalent to the condition that $M \downarrow_{PC_G(P)}$ has a direct summand isomorphic to $SC_P^{PC_G(P)}$. In this section we shall give a proof of ”if” part of the theorem by induction on $|P|$. We divide our proof in several steps.

**Lemma 3.1.** Let $Q$ be a subgroup of $P$ such that $C_{P^x}(Q) \subseteq Q$ for any $x \in G$ with $P^x \supseteq Q$. Then $M \downarrow_{N_P(Q)NC_G(Q)}$ has a direct summand isomorphic to $SC_{N_P(Q)}^{N_P(Q)NC_G(Q)}$.

**Proof.** We shall prove the lemma by induction on $|P : Q|$. If $Q = P$, then the result clearly holds. Assume that $Q \neq P$ and set $R = N_P(Q)$. Then $R \supseteq Q$. If $P^x \supseteq R$ for an element $x \in G$, then $C_{P^x}(R) \subseteq C_{P^x}(Q) \subseteq Q \subseteq R$. So $R$ satisfies the assumption in the lemma. By induction $M \downarrow_{N_P(R)NC_G(R)}$ has a direct summand isomorphic to $SC_{N_P(R)}^{N_P(R)NC_G(R)}$. As $N_P(R) \cap RC_G(R) = RC_P(R) = R$, $SC_{RC_G(R)}^R$ is a summand of $(SC_{N_P(R)}^{N_P(R)NC_G(R)}) \downarrow_{RC_G(R)}$. Thus $M \downarrow_{RC_G(R)}$ has a summand isomorphic to $SC_{RC_G(R)}^R$ and there exists an indecomposable direct summand $M_1$ of $M \downarrow_{RC_G(Q)}$ such that $M_1 \downarrow_{RC_G(R)}$ has a summand isomorphic to $SC_{RC_G(R)}^{RC_G(Q)}$. We shall show that $M_1$ is isomorphic to $SC_{RC_G(Q)}^{RC_G(Q)}$. A vertex of $M_1$ contains $R$. On the otherhand $M_1$ is $P^x \cap RC_G(Q)$-projective for some $x \in G$. Hence $P^xa \cap RC_G(Q) \supseteq R$ for some $a \in RC_G(Q)$. $P^xa \cap RC_G(Q) = RC_{P^{xa}}(Q) = R$ and therefore a vertex of $M_1$ is $R$. Set $H = RC_G(Q) \cap NC_G(Q)$. $H = R(N_G(R) \cap C_G(Q))$. We shall claim that $N_G(R) \cap C_G(Q)/(C_G(Q))$ is a $p$-group. Let $y \in N_G(R) \cap C_G(Q)$ be a $p'$-element. Then $\langle y \rangle \times Q$ acts on $R$ by conjugation and $\langle y \rangle$ centralizes $C_R(Q)$ as $C_R(Q) \subseteq Q$. 


By Thompson’s $A \times B$ Lemma (24.2 in [3]), $y$ centralizes $R$ and our claim follows. Now let $M_0$ be the Green correspondent of $M_1$ with respect to $(R, RC_G(Q), H)$. As $M_1 \downarrow_{RC_G(R)}$ has a summand isomorphic to $Sc^{RC_G}_{R}(R)$, so has $M_0 \downarrow_{RC_G(R)}$. As $M_0$ is $R$-projective and $H/RC_G(R)$ is a $p$-group, $M_0$ itself is a Scott module $Sc^H_{R}$ and therefore $M_1$ is a Scott module $Sc^{RC_G}_{R}(Q)$. □

Let $E_1$ be an abelian abelian subgroup of $P$ of maximal rank. Among the conjugates $E_1^x$ of $E_1$ with $E_1^x \subseteq P$, choose $E_0$ so that $|C_P(E_0)|$ is maximal. Set $Q_0 = C_P(E_0)$. Let $P^x \supseteq Q_0$ be a conjugate of $P$ such that $|N_{P^x}(Q_0)|$ is maximal. Now set $Q = Q_0^{−1}$ and $E = E_0^{−1}$. Then $E \subseteq P$ and $Q = C_P(E)$. In these notations we have the following.

Lemma 3.2. The following statements hold.

1. $E = \Omega_4(Q)$, that is, each element in $Q$ of order $p$ is contained in $E$.
2. $Q$ satisfies the assumption in Lemma 2.1.
3. $N_P(Q) = N_P(E)$. And if $P^x \supseteq Q$, then $|N_{P^x}(E)| \leq |N_P(Q)|$.

Proof. As $E$ is conjugate to $E_1$, $E_0$ is also of maximal rank in $P$. Hence the statement (1) follows. By our choice of $E$, $|C_P(E)| = |C_P(E_0)|$. So $|C_P(E)|$ is also maximal. If $P^x \supseteq Q$ for $x \in G$, then $P \supseteq Q^{−1}$ and $C_P(E^{−1}) \supseteq Q^{−1}$. By maximality of $|C_P(E)|$, $C_P(E^{−1}) = Q^{−1}$ and therefore $C_{P^x}(E) = Q$. Thus $C_{P^x}(Q) \subseteq C_P(Q) = Q$. Thus the statement (2) follows. $N_P(E)$ normalizes $C_P(E) = Q$ and therefore $N_P(Q) \subseteq N_P(E)$. By (1) $E$ is a characteristic subgroup of $Q$ and $N_P(Q) \subseteq N_P(E)$. If $P^x \supseteq Q$ for an element $x \in G$, then as in the above it follows that $C_{P^x}(E) = Q$ and $N_{P^x}(Q) = N_{P^x}(E)$. Now by maximality of $|N_P(Q)|$, we have that $|N_P(Q)| \geq |N_{P^x}(E)|$ and the statement (3) follows. □

For $E \subseteq P$ and $Q = C_P(E)$ chosen as in the above, $N_G(Q) \subseteq N_G(E)$ by Lemma 2.2.(1). And by Lemma 2.1 and Lemma 2.2.(2) there exists an indecomposable direct summand $M_1$ of $M_1 \downarrow_{N_G(E)}$ such that $M_1 \downarrow_{N_P(Q)C_G(Q)}$ has a direct summand isomorphic to $Sc^{N_P(Q)\cup C_G(Q)}_{N_P(Q)}$.

In the rest of this section, $E \subseteq P, Q = C_P(E)$ and the $kn_G(E)$-module $M_1$ will be those satisfying the above conditions. We have the following.

Lemma 3.3. A vertex of $M_1$ is $N_P(Q)$. $M_1 \downarrow_{C_G(E)}$ is $\{Q^x; x \in N_G(E)\}$-projective and has a direct summand isomorphic to $M^T_{Q,G(E)}$, for some simple $kn_G(E)$-module $T$ on which $C_G(Q)$ acts trivially.

Proof. A vertex of $M_1$ contains $N_P(Q)$. On the otherhand $M_1$ is $P^x \cap N_G(E)$-projective for some $x \in G$. So $P^x \cap N_G(E) \supseteq N_P(Q)$ for some $a \in N_G(E)$. Then by Lemma 2.2.(3) $P^x \cap N_G(E) = N_{P^a}(P)$ and it follows that a vertex of $M_1$ is $N_P(Q)$. For $x \in N_G(E), N_{P^a}(E) \cap C_G(E) = C_P(E)x = Q^{x}$. Hence $M_1 \downarrow_{C_G(E)}$ is $\{Q^{x}; x \in N_G(E)\}$-projective. As $M_1 \downarrow_{N_P(Q)C_G(Q)}$ has a direct summand isomorphic to $Sc^{N_P(Q)\cup C_G(Q)}_{N_P(Q)}$, there exists an indecomposable direct summand $M_0$ of $M_1 \downarrow_{C_G(E)}$ such that $M_0 \downarrow_{Q_C(G)}$ has an indecomposable direct summand isomorphic to $Sc^{Q_C(G)}_{Q_C(G)}$. 4
Such an indecomposable trivial source $kC_G(E)$-module with vertex $Q$ is isomorphic to the module described in the lemma.

In proofs of the following two lemmas we shall use the idea of Symonds in [12].

Lemma 3.4. Assume that $G = C_G(E)$. Then $H^s(G, M) \neq 0$

Proof. $C_G(P \mod E)/C_G(P)$ is a $p$-group as $E$ is central in $G$. So as a $kG/E$-module, $M$ satisfies the assumption in the theorem for $G/E$. By induction we may assume that $H^s(G/E, M) \neq 0$. We examine the Lyndon-Hochschild-Serre spectral sequence;

$$E_2^{p,q} = H^p(G/E, H^q(E, M)) \Rightarrow H^{p+q}(G, M)$$

Let $n$ be the lowest degree with $H^n(G/E, M) \neq 0$. As $E$ is central in $G$, for each $q$, a $kG/E$-module $H^q(E, M)$ is isomorphic to a direct sum of some copies of $M$ (or 0). Hence $H^n(G/E, H^q(E, M)) = 0$ for $m < n$. Thus $E_\infty^{n,0} \neq 0$ and $H^n(G, M) \neq 0$. 

By Lemma 2.3 and Lemma 2.4 $H^s(C_G(E), M) \neq 0$. Using this fact we shall examine $H^s(N_G(E), M)$ in the following two lemmas.

Let $r$ be the rank of $E$. Set $E = \langle a_1, \ldots, a_r \rangle$ and $\alpha_i \in H^1(E, k) = \text{Hom}(E,k)$ be the element dual to $a_i$. Then letting $\beta_i = \beta(\alpha_i)$ we have the polynomial subalgebra $k[\beta_1, \ldots, \beta_r]$ in $H^*(E, k)$, where $\beta$ is the Bockstein map. Using Evens’ norm map, we obtain homogeneous elements $\zeta_1, \ldots, \zeta_r \in H^s(C_G(E), k)$ such that $\text{res}_{E}^G(\zeta_i) = \beta_1^{p^s}$ where $p^s$ is the $p$-part of $|C_G(E) : E|$. Set $R = k[\zeta_1, \ldots, \zeta_r] \subseteq H^s(C_G(E), k)$ and $R_0 = \text{res}_{E}^G(R)$. The elements $\zeta_i$ can be constructed in the prime field $\mathbb{F}_p$.

We however do not know whether $R$ can be taken $N_G(E)$-invariant although $R_0$ is $N_G(E)$-invariant. We remark the following fact.

For $x \in N_G(E)$, write $\beta_i^x = \sum_{j=1}^r \lambda_{ij} \beta_j$, where $\lambda_{ij} \in \mathbb{F}_p$. Then by our choice of $\zeta_i$, we have that $\text{res}_{E}^G(\zeta_i^x - \sum_{j=1}^r \lambda_{ij} \zeta_j) = 0$. So $\text{res}_{E}^G(\zeta_i^x - \sum_{j=1}^r \lambda_{ij} \zeta_j) = 0$ is nilpotent for each $N_G(E)$-conjugate $Q^p$ because $\Omega_1(Q) = E$. So replacing $\zeta_i$’s by its suitable $p$-powers, we can assume that $\text{res}_{Q^p}^G(\zeta_i^x - \sum_{j=1}^r \lambda_{ij} \zeta_j) = 0$ for any $Q^p$. The $kN_G(E)$-module $M_1$ defined in Lemma 2.3 is $\{Q^p : y \in N_G(E)\}$-projective as $kC_G(E)$-module. Therefore for any element $\gamma \in H^s(C_G(E), M_1)$, we have $\gamma \cdot \zeta_i^x = \gamma \cdot (\sum_{j=1}^r \lambda_{ij} \zeta_j)$. Thus when we consider multiplications of the elements in $R$ on $H^s(C_G(E), M_1)$, we may assume that $R$ has an $N_G(E)$-action which coincides with that on $R_0$.

Lemma 3.5. Assume that $G = N_G(E)$. Then $\text{res}_{E}^G(\zeta_i^x \text{tr}_{E/G}(H^s(C_G(E), M))) = 0$.

Proof. By a result of Evens (Theorem 10.3.5 [7], see also [6] and [1]), $H^s(C_G(E), M)$ is free over the polynomial algebra $R$ defined in the above. Let $n$ be the lowest degree with $H^n(C_G(E), M) \neq 0$. By minimality of $n$, $H^n(C_G(E), M) \cap H^*(C_G(E), M) I = 0$, where $I$ is the ideal in $R$ of elements of positive degree. So a $k$-basis of $H^n(C_G(E), M)$ can be extended to a free $R$-basis of $H^*(C_G(E), M)$ and we can conclude that $H^n(C_G(E), M) \cdot R \cong H^n(C_G(E), M) \otimes_k R$. As is remarked in [12], $R_0$ contains a free submodule $F_0$ as $G/C_G(E)$-module. Set $F = R \cap (\text{res}_{E}^G)^{-1}(F_0)$. Then by the above remark it follows that $H^n(C_G(E), M) \cdot F \cong H^n(C_G(E), M) \otimes_k F$ is $G$-invariant and $H^n(C_G(E), M) \cdot F \cong H^n(C_G(E), M) \otimes k F_0$ as $G/C_G(E)$-modules. Thus
Theorem 3.7. \( H^*(C_G(E),M) \) also contains a free \( G/C_G(E) \)-module. So there exists an element \( \gamma \in H^*(C_G(E),M) \) such that \( 0 \neq \sum_{x \in G/C_G(E)} \gamma^x = \text{res}_{C_G(E)} G tr_{C_G(E)} G (\gamma) \).

Proof. Let \( \exists \text{ an element } \gamma \in H^*(C_G(E),M) \) such that \( 0 \neq \sum_{x \in G/C_G(E)} \gamma^x = \text{res}_{C_G(E)} G tr_{C_G(E)} G (\gamma) \).

For a subgroup \( A \subset C_G(E) \) with \( A \nsubseteq E \), take a maximal subgroup \( E_1 \) of \( E \) such that \( E_1 \supseteq A \cap E \). Using the isomorphism \( AE/A \cong E/A \cap E \) and the epimorphism \( E/A \cap E \to E/E_1 \), we have an element \( \eta(A) \in \text{Inf}(H^2(AE/A,k)) \subset H^2(AE,k) \) such that \( \text{res}_{AE}^E (\eta(A)) \in H^2(E/k) \) is not nilpotent and \( \text{res}_{AE}^E (\eta(A)) = 0 \). Using Evans’ norm map, set \( \tau(A) = \text{norm}_{AE}^{C_G(E)} (\eta(A)) \in H^*(C_G(E),k) \). By Mackey formula \( \tau(A) \) also satisfies the above conditions for \( \eta(A) \). And set \( \rho(A) = \prod_{x \in N_G(E)/C_G(E)} \tau(A)^x \in H^*(C_G(E),k) \). Finally set \( \rho = \prod_A \rho(A) \in H^*(C_G(E),k) \), where the product is taken over the set of subgroups \( A \subset C_G(E) \) with \( A \nsubseteq E \). \( \rho \) is \( N_G(E) \)-invariant. It holds that \( \text{res}_{AE}^E (\rho) = 0 \) for any subgroup \( A \subset C_G(E) \) with \( A \nsubseteq E \) and \( \text{res}_{AE}^E (\rho) \) is not nilpotent. Notice that \( \rho \) is regular on \( H^*(C_G(E),M_1) \) where \( M_1 \) is the \( kN_G(E) \)-module in Lemma 2.3 because \( E \) is central in \( C_G(E) \) and \( M_1 \) is a trivial source module with kernel containing \( E \).

Lemma 3.6. Assume that \( G = N_G(E) \). Then there exists an element \( \alpha \in H^*(G,M) \) such that \( \text{res}_{G}^Q (\alpha) \neq 0 \) and \( \text{res}_{A}^Q (\alpha) = 0 \) for any subgroup \( A \subset G \) with \( A \nsubseteq E \).

Proof. Set \( C = C_G(E) \). By Lemma 2.5 there exists \( \gamma \in H^*(C,M) \) such that \( 0 \neq \text{res}_{C}^G (\gamma) \). Set \( \alpha = \text{tr}_{C}^G (\gamma \cdot \rho) \in H^*(G,M) \). We shall show that \( \alpha \) satisfies the assumptions in the lemma.

For a subgroup \( A \) of \( G \), \( \text{res}_{A}^C (\alpha) = \text{res}_{A}^G \text{tr}_{C}^G (\gamma \cdot \rho) = \sum_{x \in C \cap G/A} \text{tr}_{C \cap A}^A \text{res}_{C \cap A}^C (\gamma \cdot \rho)^x \). As \( \rho \) is \( G \)-invariant, \( \text{res}_{C \cap A}^C (\gamma \cdot \rho)^x = \text{res}_{C \cap A}^C (\gamma^x) \text{res}_{C \cap A}^C (\rho) \). If \( A \nsubseteq E \), then \( C \cap A \nsubseteq E \) and therefore \( \text{res}_{A}^C (\alpha) = 0 \). Again by the fact that \( \rho \) is \( G \)-invariant \( \text{res}_{A}^G (\alpha) = \text{res}_{A}^G \text{tr}_{C}^G (\gamma \cdot \rho) = \text{res}_{A}^C \text{tr}_{C}^G (\gamma) \cdot \rho \neq 0 \) because \( \rho \) is regular on \( H^*(C,M) \).

Now we can complete a proof for "If" part of the theorem of Symonds.

Theorem 3.7. If \( C_G(P) \) acts trivially on \( S \), then \( H^*(G,M_{P,S}) \neq 0 \).

Proof. Let \( M_1 \) be the \( kN_G(E) \)-module in Lemma 2.3. Then by Lemma 2.6, there exists an element \( \alpha \in H^*(N_G(E),M_1) \) such that \( \text{res}_{Q}^N (\alpha) \neq 0 \) and \( \text{res}_{A}^N (\alpha) = 0 \) for any subgroup \( A \subset N_G(E) \) with \( A \nsubseteq E \). As \( M_1 \) is a direct summand of \( M_{N_G(E)} \), we can regard \( \alpha \in H^*(N_G(E),M) \) for which the same conditions as in the above hold. We shall show that \( \text{res}_{Q}^G \text{tr}_{N_G(E)} G (\alpha) \neq 0 \). For an element \( x \in G \), if \( N_G(E) \cap Q^x \supseteq E \), then \( E^x = E \) as \( \Omega_1(Q) = E \) and hence \( x \in N_G(E) \). Thus for \( x \notin N_G(E) \), we have that \( \text{res}_{Q}^G E^x \alpha = (\text{res}_{N_G(E)}^N E^x \alpha) = (\text{res}_{N_G(E)}^N Q^{x-1} \alpha)^x = 0 \). Now Mackey formula says that \( \text{res}_{Q}^G \text{tr}_{N_G(E)} G (\alpha) = \text{res}_{Q}^N (\alpha) \neq 0 \).

References

[8] A. Hida, Control of fusion and cohomology of finite groups, RIMS 講究録「有限群のコホモロジー論とその周辺」本号