ON A THEOREM OF MISLIN ON COHOMOLOGY ISOMORPHISM AND CONTROL OF FUSION

北海道教育大学旭川校　奥山　哲郎 (TETSURO OKUYAMA)

HOKKAIDO UNIVERSITY OF EDUCATION, ASAHIKAWA CAMPUS

Introduction

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$. In 1990 [9] G.Mislin proved the following remarkable theorem.

Theorem (Mislin). *Let H be a subgroup of G. Then the restriction map in mod-p cohomology $\mathrm{res}^G_H : H^*(G,k) \to H^*(H,k)$ is an isomorphism if and only if H controls strong p-fusion in G.*

"If" part in the theorem has long been known to be true. For "Only if"part Mislin's proof uses deep results from algebraic topology. In 2001 [11] V.P.Snaith gave an alternating proof of Mislin's theorem which uses also topological results. In [10] G.R.Robinson remarked that Mislin's theorem can be obtained if one could prove the non-vanishing of cohomology of certain types of trivial source kG-modules.

Isomorphism classes of indecomposable trivial source kG-modules are parametralized as follows. Let P be a p-subgroup of G and S be a simple $kN_G(P)$-module. Let $M^{N_G(P)}_{P,S}$ be a projective cover of S as $kN_G(P)/P$-module. Inflating $M^{N_G(P)}_{P,S}$ to $kN_G(P)$ and taking its Green correspondent, we obtain an indecomposable trivial source module $M^G_{P,S}$ with vertex P. And each indecomposable trivial source module is obtained in this way.

P.Symonds in [13] proved the following result from which Mislin's theorem is obtained following Robinson's remark.

Theorem (Symonds). *In the notations above, $H^*(G, M^G_{P,S}) \neq 0$ if and only if $C_G(P)$ acts trivially on S.*

A proof of the above theorem given by P.Symonds needs also topological methods. My aim in this talk is to give an algebraic proof of the theorem of P.Symonds.

A.Hida [8] also obtained an algebraic proof of the above Symonds'theorem and explained his idea in his talk at this meeting. A very elegant proof !!

In my lecture I first introduced the idea of Robinson to find an algebraic proof of Mislin's theorem and how his idea relates Symonds'theorem. This is included in section 1 in this note. And then I discussed the theorem of Symonds. In the lecture I only gave an outline of my proof of the theorem. I shall give my proof in detail in this note.

"Only if"part of the theorem has been essentially proved by Benson, Carlson and Robinson in [5]. In section 2 in this note we shall give a proof of "Only if part" following arguments by them.

For "If"part we first reduce the problem to some p-local subgroup. This is done in section 3. Our p-local subgroup is a normalizer of some elementary abelian p-group. Then we use the idea of Symonds in [12] to find a nonzero cohomology element. There he made use of the Lyndon-Hochschild-Serre spectral sequence and some result on the action of $\mathrm{Aut}(E)$ on the cohomology algebra $H^*(E,k)$, where E is an elementary abelian p-group. He needed also a result of Duflot [6] on the depth of cohomology algebras of groups with central elementary abelian groups. For these results there has been given algebraic proofs (see for example [2],[4] and [7]) and we believe that our proof of the theorem is an algebraic one.

1. Robinson's Idea

In this section let H be a subgroup of G and assume that $\mathrm{res}^G_H : H^*(G,k) \to H^*(H,k)$ is an isomorphsm. We first remark the following.

Lemma 1.1. *H contains a Sylow p-subgroup of G.*

Proof. Consider an H-injective hull of k_G; $0 \to k_G \xrightarrow{f} k_H \uparrow^G \to L \to 0$. We obtain the following long exact sequence

$$\to H^n(G,k) \xrightarrow{f_*} H^n(G, k_H \uparrow^G) \to H^n(G,L) \to H^{n+1}(G,k) \xrightarrow{f_*} H^{n+1}(G, k_H \uparrow^G) \to$$

Identify $H^n(G, k_H \uparrow^G)$ with $H^n(H, k_H)$ by Eckmann-Shapiro. Then it follows that the map f_* coincides with the restriction map res^G_H. By our assumption we have $H^n(G,L)$ for $n \geqq 0$. By a theorem of Benson- Carlson-Robinson (Theorem 2.4 [5]), $\hat{H}^n(G,L) = 0$ for all n, where $\hat{H}^n$ is Tate's cohomology. In particular, $\mathrm{res}^G_H : \hat{H}^{-1}(G,k_G) \to \hat{H}^{-1}(H,k_H)$ is an isomorphism. Any non zero element in $\hat{H}^{-1}(G,k)$ represents the almost split sequence terminating at k_G and it is well known that the sequence does not split as a sequence of kH-modules if and only if H contains a Sylow p-subgroup of G. Thus the lemma is proved. $\square$

Assume that H contains a Sylow p-subgroup of G and H does not control p-fusion. Then there exists a p-subgroup P of H such that $N_G(P) \supsetneqq C_G(P)N_H(P)$. Choose P maximal with this property, then $C_G(P) = Z(P) \times O_{p'}(C_G(P))$ and $C_G(P)N_H(P)/P$ is a strongly p-embedded subgroup of $N_G(P)/P$. Set $C = C_G(P)N_H(P)$. Then $k_C \uparrow^{N_G(P)} = k \oplus M$ for some $kN_G(P)$-module M. Each indecomposable summand of M has the form $M^{N_G(P)}_{P,S}$ with $C_G(P) \subset \mathrm{Ker}\, S$. $(k_H \uparrow^G) \downarrow_{N_G(P)} = k_{N_H(P)} \uparrow^{N_G(P)} \oplus U = k_C \uparrow^{N_G(P)} \oplus U' = k_G \oplus M \oplus U'$ for some $kN_G(P)$-modules U, U'. By a theorem of Burry-Carlson, $k_H \uparrow^G = k_G \oplus M^G_{P,S} \oplus V$ with $\mathrm{Ker}\, S \supset C_G(P)$.

Now Symonds'theorem implies that $H^(G, M^G_{P,S}) \neq 0$ and we can conclude that $H^*(H,k) = H^*(G, k_H \uparrow^G) \supsetneqq H^*(G,k)$ and the "only if" part of Mislin's theorem follows.

2. Proof of "Only if" Part

Let P be a p-subgroup of G and S be a simple $kN_G(P)$-module. And let $M^G_{P,S}$ be an indecomposable kG-module with vertex P and with trivial source described in introduction. In these notations we shall prove the following.

Theorem 2.1. *$H^*(G, M^G_{P,S}) = 0$ if $C_G(P)$ acts nontrivially on S.*

We argue following a proof of Proposition 5.3 in [5]. If $C_G(P)$ acts nontrivially on S, then there exists a p'-elemnt $y \neq 1$ in $C_G(P)$ such that y acts nontrivially on S. Thus there exists a one dimensional submodule M_0 of $S{\downarrow}_{\langle y\rangle\times P}$ on which y acts nontrivially. Then $M_0{\uparrow}^{N_G(P)}$ has a summand isomorphic to $M^{N_G(P)}_{P,S}$ because $M_0{\uparrow}^{N_G(P)}$ is a projective $kN_G(P)/P$-module and $\mathrm{Hom}_{kN_G(P)}(M_0{\uparrow}^{N_G(P)}, S) \cong \mathrm{Hom}_{k\langle y\rangle\times P}(M_0, S{\downarrow}_{\langle y\rangle\times P}) \neq 0$. Therefore $M = M^G_{P,S}$ appears in summand of $M_0{\uparrow}^G$ and $H^*(G, M) \leq H^*(G, M_0{\uparrow}^G)$. Now the result follows by Lemma 5.1 in [5].

3. Proof of "If" Part

Let H be a subgroup of G and P be a p-subgroup of H. Then the module $M^H_{P,k}$ where $k = k_{N_H(P)}$ is the trivial $kN_H(P)$-module is called a Scott module of H with vertex P and we shall denote it by Sc^H_P. It is well known that Sc^H_P is a unique trivial source module of H with vertex P which contains k_H.

Throughout this section let $M = M^G_{P,S}$ where P is a p-subgroup of G and S is a simple $kN_G(P)$-module on which $C_G(P)$ acts trivially. Notice that the condition that $C_G(P)$ acts trivially on S is equivalent to the condition that $M{\downarrow}_{PC_G(P)}$ has a direct summand isomorphic to $Sc^{PC_G(P)}_P$. In this section we shall give a proof of "if"part of the theorem by induction on $|P|$. We divide our proof in several steps.

Lemma 3.1. *Let Q be a subgroup of P such that $C_{P^x}(Q) \subseteq Q$ for any $x \in G$ with $P^x \supseteq Q$. Then $M{\downarrow}_{N_P(Q)C_G(Q)}$ has a direct summand isomorphic to $Sc^{N_P(Q)C_G(Q)}_{N_P(Q)}$.*

Proof. We shall prove the lemma by induction on $[P : Q]$. If $Q = P$, then the result clearly holds. Assume that $Q \neq P$ and set $R = N_P(Q)$. Then $R \supsetneq Q$. If $P^x \supseteq R$ for an element $x \in G$, then $C_{P^x}(R) \subseteq C_{P^x}(Q) \subseteq Q \subset R$. So R satisfies the assumption in the lemma. By induction $M{\downarrow}_{N_P(R)C_G(R)}$ has a direct summand isomorphic to $Sc^{N_P(R)C_G(R)}_{N_P(R)}$. As $N_P(R) \cap RC_G(R) = RC_P(R) = R$, $Sc^{RC_G(R)}_R$ is a summand of $(Sc^{N_P(R)C_G(R)}_{N_P(R)}){\downarrow}_{RC_G(R)}$. Thus $M{\downarrow}_{RC_G(R)}$ has a summand isomorphic to $Sc^{RC_G(R)}_R$ and there exists an indecomposable direct summand M_1 of $M{\downarrow}_{RC_G(Q)}$ such that $M_1{\downarrow}_{RC_G(R)}$ has a summand isomorphic to $Sc^{RC_G(R)}_R$. We shall show that M_1 is isomorphic to $Sc^{RC_G(Q)}_R$. A vertex of M_1 contains R. On the otherhand M_1 is $P^x \cap RC_G(Q)$-projective for some $x \in G$. Hence $P^{xa} \cap RC_G(Q) \supseteq R$ for some $a \in RC_G(Q)$. $P^{xa} \cap RC_G(Q) = RC_{P^{xa}}(Q) = R$ and therefore a vertex of M_1 is R. Set $H = RC_G(Q) \cap N_G(R)$. $H = R(N_G(R) \cap C_G(Q))$. We shall claim that $N_G(R) \cap C_G(Q)/C_G(R)$ is a p-group. Let $y \in N_G(R) \cap C_G(Q)$ be a p'-element. Then $\langle y\rangle \times Q$ acts on R by conjugation and $\langle y\rangle$ centralizes $C_R(Q)$ as $C_R(Q) \subseteq Q$.

By Thompson's $A \times B$ Lemma (24.2 in [3]), y centralizes R and our claim follows. Now let M_0 be the Green correspondent of M_1 with respect to $(R, RC_G(Q), H)$. As $M_1{\downarrow}_{RC_G(R)}$ has a summand isomorphic to $Sc_R^{RC_G(R)}$, so has $M_0{\downarrow}_{RC_G(R)}$. As M_0 is R-projective and $H/RC_G(R)$ is a p-group, M_0 itself is a Scott module Sc_R^H and therefore M_1 is a Scott module $Sc_R^{RC_G(Q)}$. □

Let E_1 be an elementary abelian subgroup of P of maximal rank. Among the conjugates E_1^x of E_1 with $E_1^x \subseteq P$, choose E_0 so that $|C_P(E_0)|$ is maximal. Set $Q_0 = C_P(E_0)$. Let $P^a \supseteq Q_0$ be a conjugate of P such that $|N_{P^a}(Q_0)|$ is maximal. Now set $Q = Q_0^{a^{-1}}$ and $E = E_0^{a^{-1}}$. Then $E \subseteq P$ and $Q = C_P(E)$. In these notations we have the following.

Lemma 3.2. *The following statements hold.*

1. $E = \Omega_1(Q)$*, that is, each element in* Q *of order* p *is contained in* E*.*
2. Q *satisfies the assumption in Lemma 2.1.*
3. $N_P(Q) = N_P(E)$*. And if* $P^x \supseteq Q$*, then* $|N_{P^x}(E)| \le |N_P(Q)|$*.*

Proof. As E is conjugate to E_1, E is also of maximal rank in P. Hence the statement (1) follows. By our choice of E, $|C_P(E)| = |C_P(E_0)|$. So $|C_P(E)|$ is also maximal. If $P^x \supseteq Q$ for $x \in G$, then $P \supseteq Q^{x^{-1}}$ and $C_P(E^{x^{-1}}) \supseteq Q^{x^{-1}}$. By maximality of $|C_P(E)|$, $C_P(E^{x^{-1}}) = Q^{x^{-1}}$ and therefore $C_{P^x}(E) = Q$. Thus $C_{P^x}(Q) \subseteq C_{P^x}(E) = Q$. Thus the statement (2) follows. $N_P(E)$ normalizes $C_P(E) = Q$ and therefore $N_P(E) \subseteq N_P(Q)$. By (1) E is a characteristic subgroup of Q and $N_P(Q) \subseteq N_P(E)$. If $P^x \supseteq Q$ for an element $x \in G$, then as in the above it follows that $C_{P^x}(E) = Q$ and $N_{P^x}(Q) = N_{P^x}(E)$. Now by maximality of $|N_P(Q)|$, we have that $|N_P(Q)| \ge |N_{P^x}(Q)| = |N_{P^x}(E)|$ and the statement (3) follows. □

For $E \subseteq P$ and $Q = C_P(E)$ chosen as in the above, $N_G(Q) \subseteq N_G(E)$ by Lemma 2.2.(1). And by Lemma 2.1 and Lemma 2.2.(2) there exists an indecomposable direct summand M_1 of $M{\downarrow}_{N_G(E)}$ such that $M_1{\downarrow}_{N_P(Q)C_G(Q)}$ has a direct summand isomorphic to $Sc_{N_P(Q)}^{N_P(Q)C_G(Q)}$.

In the rest of this section , $E \subseteq P, Q = C_P(E)$ and the $kN_G(E)$-module M_1 will be those satifying the above conditions. We have the following.

Lemma 3.3. *A vertex of* M_1 *is* $N_P(Q)$*.* $M_1{\downarrow}_{C_G(E)}$ *is* $\{Q^x; x \in N_G(E)\}$*-projective and has a direct summand isomorphic to* $M_{Q,T}^{C_G(E)}$*, for some simple* $kN_{C_G(E)}(Q)$*-module* T *on which* $C_G(Q)$ *acts trivially.*

Proof. A vertex of M_1 contains $N_P(Q)$. On the otherhand M_1 is $P^x \cap N_G(E)$-projective for some $x \in G$. So $P^{xa} \cap N_G(E) \supseteq N_P(Q)$ for some $a \in N_G(E)$. Then by Lemma 2.2.(3) $P^{xa} \cap N_G(E) = N_{P^{xa}}(E) = N_P(Q)$ and it follows that a vertex of M_1 is $N_P(Q)$. For $x \in N_G(E), N_P(Q)^x \cap C_G(E) = C_P(E)^x = Q^x$. Hence $M_1{\downarrow}_{C_G(E)}$ is $\{Q^x; x \in N_G(E)\}$-projective. As $M_1{\downarrow}_{N_P(Q)C_G(Q)}$ has a direct summand isomorphic to $Sc_{N_P(Q)}^{N_P(Q)C_G(Q)}$, there exists an indecomposable direct summand M_0 of $M_1{\downarrow}_{C_G(E)}$ such that $M_0{\downarrow}_{QC_G(Q)}$ has an indecomposable direct summand isomorphic to $Sc_Q^{QC_G(Q)}$.

Such an indecomposable trivial source $kC_G(E)$-module with vertex Q is isomorphic to the module described in the lemma. $\square$

In proofs of the following two lemmas we shall use the idea of Symonds in [12].

Lemma 3.4. *Assume that $G = C_G(E)$. Then $H^*(G, M) \neq 0$*

Proof. $C_G(P \bmod E)/C_G(P)$ is a p-group as E is central in G. So as a kG/E-module, M satisfies the assumption in the theorem for G/E. By induction we may assume that $H^*(G/E, M) \neq 0$. We examine the Lyndon-Hochschild-Serre spectral sequence ;

$$E_2^{p,q} = H^p(G/E, H^q(E, M)) \Rightarrow H^{p+q}(G, M)$$

Let n be the lowest degree with $H^n(G/E, M) \neq 0$. As E is central in G, for each q, a kG/E-module $H^q(E, M)$ is isomorphic to a direct sum of some copies of M (or 0). Hence $H^m(G/E, H^q(E, M)) = 0$ for $m < n$. Thus $E_\infty^{n,0} \neq 0$ and $H^n(G, M) \neq 0$. $\square$

By Lemma 2.3 and Lemma 2.4 $H^*(C_G(E), M) \neq 0$. Using this fact we shall examine $H^*(N_G(E), M)$ in the following two lemmas.

Let r be the rank of E. Set $E = \langle a_1, \cdots, a_r \rangle$ and $\alpha_i \in H^1(E, k) = \mathrm{Hom}(E, k)$ be the element dual to a_i. Then letting $\beta_i = \beta(\alpha_i)$ we have the polynomial subalgebra $k[\beta_1, \cdots, \beta_r]$ in $H^*(E, k)$, where β is the Bockstein map. Using Evens' norm map, we obtain homogeneous elements $\zeta_1, \cdots, \zeta_r \in H^*(C_G(E), k)$ such that $\mathrm{res}_E^{C_G(E)}(\zeta_i) = {\beta_i}^{p^a}$ where p^a is the p-part of $|C_G(E) : E|$. Set $R = k[\zeta_1, \cdots, \zeta_r] \subseteq H^*(C_G(E), k)$ and $R_0 = \mathrm{res}_E^{C_G(E)}(R)$. The elements ζ_i can be constructed in the prime field $\mathbb{F}_p$. We however do not know whether R can be taken $N_G(E)$-invariant although R_0 is $N_G(E)$-invariant. We remark the following fact.

For $x \in N_G(E)$, write $\beta_i^x = \sum_{j=1}^r \lambda_{ij}\beta_j$, where $\lambda_{ij} \in \mathbb{F}_p$. Then by our choice of ζ_i, we have that $\mathrm{res}_E^{C_G(E)}(\zeta_i^x - \sum_{j=1}^r \lambda_{ij}\zeta_j) = 0$. So $\mathrm{res}_{Q^y}^{C_G(E)}(\zeta_i^x - \sum_{j=1}^r \lambda_{ij}\zeta_j)$ is nilpotent for each $N_G(E)$-congugate Q^y because $\Omega_1(Q) = E$. So replacing ζ_i's by its suitable p-powers, we can assume that $\mathrm{res}_{Q^y}^{C_G(E)}(\zeta_i^x - \sum_{j=1}^r \lambda_{ij}\zeta_j) = 0$ for any Q^y. The $kN_G(E)$-module M_1 defined in Lemma 2.3 is $\{Q^y; y \in N_G(E)\}$-projective as $kC_G(E)$-module. Therefore for any element $\gamma \in H^*(C_G(E), M_1)$, we have $\gamma \cdot \zeta_i^x = \gamma \cdot (\sum_{j=1}^r \lambda_{ij}\zeta_j)$. Thus when we consider multiplications of the elements in R on $H^*(C_G(E), M_1)$,we may assume that R has an $N_G(E)$-action which coincides with that on R_0.

Lemma 3.5. *Assume that $G = N_G(E)$. Then $\mathrm{res}_{C_G(E)}^G \mathrm{tr}_{C_G(E)}^G(H^*(C_G(E), M)) \neq 0$.*

Proof. By a result of Evens (Theorem 10.3.5 [7], see also [6] and [1]), $H^*(C_G(E), M)$ is free over the polynomial algebra R defined in the above. Let n be the lowest degree with $H^n(C_G(E), M) \neq 0$. By minimality of n, $H^n(C_G(E), M) \cap H^*(C_G(E), M)I = 0$, where I is the ideal in R of elements of positive degree. So a k-basis of $H^n(C_G(E), M)$ can be extended to a free R-basis of $H^*(C_G(E), M)$ and we can conclude that $H^n(C_G(E), M) \cdot R \cong H^n(C_G(E), M) \otimes_k R$. As is remarked in [12], R_0 contains a free submodule F_0 as $G/C_G(E)$-module. Set $F = R \cap (\mathrm{res}_E^{C_G(E)})^{-1}(F_0)$. Then by the above remark it follows that that $H^n(C_G(E), M) \cdot F \cong H^n(C_G(E), M) \otimes_k F$ is G-invariant and $H^n(C_G(E), M) \cdot F \cong H^n(C_G(E), M) \otimes_k F_0$ as $G/C_G(E)$-modules. Thus

$H^*(C_G(E), M)$ also contains a free $G/C_G(E)$-module. So there exists an element $\gamma \in H^*(C_G(E), M)$ such that $0 \neq \sum_{x \in G/C_G(E)} \gamma^x = \mathrm{res}^G_{C_G(E)} \mathrm{tr}^G_{C_G(E)}(\gamma)$. □

For a subgroup $A \subset C_G(E)$ with $A \not\supseteq E$, take a maximal subgroup E_1 of E such that $E_1 \supseteq A \cap E$. Using the isomorphism $AE/A \cong E/A \cap E$ and the epimorphism $E/A \cap E \to E/E_1$, we have an element $\eta(A) \in \mathrm{Inf}(H^2(AE/A, k)) \subset H^2(AE, k)$ such that $\mathrm{res}^{AE}_E(\eta(A)) \in H^2(E, k)$ is not nilpotent and $\mathrm{res}^{AE}_A(\eta(A)) = 0$. Using Evens' norm map, set $\tau(A) = \mathrm{norm}^{C_G(E)}_{AE}(\eta(A)) \in H^*(C_G(E), k)$. By Mackey formula $\tau(A)$ also satisfies the above conditions for $\eta(A)$. And set $\rho(A) = \prod_{x \in N_G(E)/C_G(E)} \tau(A)^x \in H^*(C_G(E), k)$. Finally set $\rho = \prod_A \rho(A) \in H^*(C_G(E), k)$, where the product is taken over the set of subgroups A of $C_G(E)$ with $A \not\supseteq E$. ρ is $N_G(E)$-invariant. It holds that $\mathrm{res}^{C_G(E)}_A(\rho) = 0$ for any subgroup $A \subset C_G(E)$ with $A \not\supseteq E$ and $\mathrm{res}^{C_G(E)}_E(\rho) \in H^*(E, k)$ is not nilpotent. Notice that ρ is regular on $H^*(C_G(E), M_1)$ where M_1 is the $kN_G(E)$-module in Lemma 2.3 because E is central in $C_G(E)$ and M_1 is a trivial source module with kernel containing E.

Lemma 3.6. *Assume that $G = N_G(E)$. Then there exists an element $\alpha \in H^*(G, M)$ such that $\mathrm{res}^G_Q(\alpha) \neq 0$ and $\mathrm{res}^G_A(\alpha) = 0$ for any subgroup $A \subset G$ with $A \not\supseteq E$.*

Proof. Set $C = C_G(E)$. By Lemma 2.5 there exists $\gamma \in H^*(C, M)$ such that $0 \neq \mathrm{res}^G_C \mathrm{tr}^G_C(\gamma)$. Set $\alpha = \mathrm{tr}^G_C(\gamma \cdot \rho) \in H^*(G, M)$. We shall show that α satisfies the assumptions in the lemma.

For a subgroup A of G , $\mathrm{res}^G_A(\alpha) = \mathrm{res}^G_A \mathrm{tr}^G_C(\gamma \cdot \rho) = \sum_{x \in C\backslash G/A} \mathrm{tr}^A_{C \cap A} \mathrm{res}^C_{C \cap A}((\gamma \cdot \rho)^x)$. As ρ is G-invariant, $\mathrm{res}^C_{C \cap A}((\gamma \cdot \rho)^x) = \mathrm{res}^C_{C \cap A}(\gamma^x) \mathrm{res}^C_{C \cap A}(\rho)$. If $A \not\supseteq E$, then $C \cap A \not\supseteq E$ and therefore $\mathrm{res}^G_A(\alpha) = 0$. Again by the fact that ρ is G-invariant $\mathrm{res}^G_C(\alpha) = \mathrm{res}^G_C \mathrm{tr}^G_C(\gamma \cdot \rho) = (\mathrm{res}^G_C \mathrm{tr}^G_C(\gamma)) \cdot \rho \neq 0$ because ρ is regular on $H^*(C, M)$. If $\mathrm{res}^G_Q(\alpha) = 0$, then $\mathrm{res}^G_{Q^x}(\alpha) = 0$ for all $x \in G$. Then as $M{\downarrow_C}$ is $\{Q^x; x \in G\}$-projective, it follows that $\mathrm{res}^G_C(\alpha) \neq 0$ which is not the case. □

Now we can complete a proof for "If"part of the theorem of Symonds.

Theorem 3.7. *If $C_G(P)$ acts trivially on S, then $H^*(G, M^G_{P,S}) \neq 0$.*

Proof. Let M_1 be the $kN_G(E)$-module in Lemma 2.3. Then by Lemma 2.6, there exists an element $\alpha \in H^*(N_G(E), M_1)$ such that $\mathrm{res}^{N_G(E)}_Q(\alpha) \neq 0$ and $\mathrm{res}^{N_G(E)}_A(\alpha) = 0$ for any subgroup $A \subset N_G(E)$ with $A \not\supseteq E$. As M_1 is a direct summand of $M{\downarrow_{N_G(E)}}$, we can regard $\alpha \in H^*(N_G(E), M)$ for which the same conditions as in the above hold. We shall show that $\mathrm{res}^G_Q \mathrm{tr}^G_{N_G(E)}(\alpha) \neq 0$. For an element $x \in G$, if $N_G(E) \cap Q^x \supseteq E$, then $E^x = E$ as $\Omega_1(Q) = E$ and hence $x \in N_G(E)$. Thus for $x \notin N_G(E)$, we have that $\mathrm{res}^{N_G(E)^x}_{N_G(E)^x \cap Q}(\alpha^x) = (\mathrm{res}^{N_G(E)}_{N_G(E) \cap Q^{x^{-1}}}(\alpha))^x = 0$. Now Mackey formula says that $\mathrm{res}^G_Q \mathrm{tr}^G_{N_G(E)}(\alpha) = \mathrm{res}^{N_G(E)}_Q(\alpha) \neq 0$. □

References

[1] A.Adem and R.J.Milgram, *The mod 2 cohomology rings of rank 3 simple groups are Cohen-Macaulay*, Ann. of Math. Stud., 138, 2–13, 1996

[2] J.L.Alperin, Local Representatiom Theory , Cambridge Studies in Advanced Mathematics, 11, Cambridge University Press, 1986

[3] M.Aschbacher, Finite Group Theory, (second edition), Cambridge Studies in Advanced Mathematics, 10, Cambridge University Press, 2000

[4] D.J.Benson, Representations and Cohomology II, Cambridge Studies in Advanced Mathematics, 31, Cambridge University Press, 1991

[5] D.J.Benson,J.F.Carlson and G.R.Robinson, *On the vanishing of group cohomology*, J. Algebra, 131, 40–73, 1990

[6] J.Duflot, *Depth and equivariant cohomology*, Comment. Math. Helvetici, 56, 627–637, 1981

[7] L.Evens, The Cohomology of Groups, Oxford Mathematical Monographs, Clarendon Press, 1991

[8] A.Hida, *Control of fusion and cohomology of finite groups*, RIMS 講究録「有限群のコホモロジー論とその周辺」本号

[9] G.Mislin, *On group homomorphisms inducing mod-p cohomology isomorphisms*, Comment. Math. Helvetici, 65, 454–461, 1990

[10] G.R.Robinson, *Arithmetical properties of blocks*, in Algebraic Groups and their Representations (Carter R.W and Saxl J. eds.), 213–232, Kluwer, 1998

[11] V.P.Snaith, *On a theorem of Mislin*, Bull. London Math.Soc., 33,275–278, 2001

[12] P.Symonds, *The action of automorphisms on the cohomology of a p-group*, Math. Proc. Camb. Phil. Soc., 127, 495–496, 1999

[13] P.Symonds, *Mackey functors and control of fusion*, preprint, Dec. 2002