On almost translation invariant sets of reals

Hiroshi Fujita
Ehime University (Matsuyama)

March 30, 2006

Abstract

Let $D_{\mathcal{C}}$ be the class of subsets of $\mathbb{R}$ that is uncountable, co-uncountable and translation invariant modulo countable sets. In this paper we prove that no Borel sets belongs to $D_{\mathcal{C}}$ (Theorem 3); no $\mathbf{\Delta}^1_2$ sets belongs to $D_{\mathcal{C}}$ if either every $\mathbf{\Delta}^1_2$ set has the property of Baire or every $\mathbf{\Delta}^1_2$ set is Lebesgue measurable (Theorem 4); and there is a Δ^1_2 set in $D_{\mathcal{C}}$ if every real number is constructible (Theorem 2).

Introduction

The continuum hypothesis (CH) has a variety of consequences on the structure of real line. The book “L'hypothèse du Continu” ([6]) by W.Sierpiński is a large collection of bizarre consequences of CH that seem not to follow from the usual ZFC axioms. In this article, we study propositions $\mathbf{C}_{68}$–$\mathbf{C}_{70}$ in [6] from viewpoint of descriptive set theory. These propositions concern with subsets of the real line which are translation invariant except countable sets. Before stating the propositions, we should give some definitions.

Throughout the article, $\mathbb{R}$ means the real line. For $A \subset \mathbb{R}$ and $t \in \mathbb{R}$, $A + t$ is the set of $a + t$ with $a \in A$. The set $A - t$ is defined similarly.

We say a set $A \subset \mathbb{R}$ is *translation invariant modulo countable sets* if for every $t \in \mathbb{R}$ the symmetric difference $(A + t) \triangle A$ is countable. This is clearly the case when either A or $\mathbb{R} \setminus A$ is countable. If there is any other set that is translation invariant modulo countable sets, let us call it a *$D_{\mathcal{C}}$ set.* By definition

$$D_{\mathcal{C}} = \{ A \subset \mathbb{R} : |A| > \omega,\ |\mathbb{R} \setminus A| > \omega,\ (\forall t \in \mathbb{R})[\, |(A + t) \triangle A| \leq \omega \,] \}.$$

Now we can state Sierpiński's propositions as follows: CH implies that $D_{\mathcal{C}}$ is not empty; it contains a meager set ($\mathbf{C}_{68}$), a null set ($\mathbf{C}_{69}$), and a non-measurable set ($\mathbf{C}_{70}$). The main results of this article says that there are no Borel sets in $D_{\mathcal{C}}$ (Theorem 3) and that whether some $\mathbf{\Delta}^1_2$ sets belong to $D_{\mathcal{C}}$ is independent of ZFC+CH(Theorems 2 and 4).

1 Continuum Hypothesis and $D_{\mathcal{C}}$

First of all, we give the proof of the following

Theorem 1. *The class $D_{\mathcal{C}}$ is not empty if and only if CH holds.*

PROOF: Assume CH. Then there exists an ω_1-sequence $\langle G_\alpha : \alpha < \omega_1 \rangle$ such that

(1) each G_α is a countable subgroup of $\langle \mathbb{R}, + \rangle$;

(2) $G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset G_{\alpha+1} \subset \cdots$;

(3) $G_{\alpha+1}$ contains at least 3 congruence classes modulo G_α; and

(4) $\mathbb{R} = \bigcup_{\alpha<\omega_1} G_\alpha$.

For each α, choose c_α to be a member of $G_{\alpha+1}$ that is not in G_α. Then let $A = \bigcup_{\alpha<\omega_1}(G_\alpha + c_\alpha)$. It is now easy to see that $A \in D_{\mathcal{C}}$.

For the converse, let $A \in D_{\mathcal{C}}$. Pick ω_1 many distinct members, say x_ξ ($\xi < \omega_1$), from A. For every $t \in \mathbb{R}$, there is a ξ such that $x_\xi + t \in A$, since otherwise $(A + t) \setminus A$ would contain uncountably many members of the form $x_\xi + t$. From this it follows that $\mathbb{R} = \bigcup_{\xi<\omega_1}(A - x_\xi)$. Therefore

$$\mathbb{R} \setminus A = \bigcup_{\xi<\omega_1} ((A - x_\xi) \setminus A).$$

Being the union of ω_1 many countable sets, the right hand side has size at most ω_1. Similar argument shows A has size at most ω_1. ■

The "if" part of Theorem 1 is due to Sierpiński. It has first appeared in [7]. Even earlier, S.Banach has shown similar result using the circle instead of the line ([1]). The proof of "only if" part of the theorem resembles that of the Rothberger theorem (that if both Lusin and Sierpiński sets exist then CH holds). This is why we suspect that Sierpiński knew it. But as far as we know, Sierpiński did not write about it. The equivalence has been pointed out by J.Shinoda in [5, Section 3]. M.Laczkovich's paper [3] contains related results in more general context. We do not know who was the first to publish.

Next we show, assuming the axiom of constructibility ($V = L$), the construction described in Theorem 1 can be carried out in a Σ_1 way over the class of hereditarily countable sets, so that the resulting set A and its complement are both Σ^1_2.

From here to the end of this section, we assume $V = L$. Define δ_α for each $\alpha < \omega_1$ as follows:

(5) $\delta_0 = \omega + \omega$;

(6) $\delta_{\alpha+1}$ is the smallest limit δ such that $L_\delta \models$ *"L_{δ_α} is countable"*; and

(7) $\delta_\lambda = \sup_{\alpha<\lambda} \delta_\alpha$ for limit λ.

Then the sequence $\langle \delta_\alpha : \alpha < \omega_1 \rangle$ is Δ_1 over L_{ω_1}. Let $G_\alpha = \mathbb{R} \cap L_{\delta_\alpha}$. It is easy to see that $\langle G_\alpha : \alpha < \omega_1 \rangle$ meets four clauses in the proof of Theorem 1. We choose c_α to be the $<_L$-smallest member of $G_{\alpha+1} \setminus G_\alpha$, where $<_L$ denotes the canonical wellordering of the constructible universe. The statement "$x = c_\alpha$" is Δ_1 over L_{ω_1} since, in L_{ω_1},

$$\begin{aligned} x = c_\alpha \iff & x \in \mathbb{R} \wedge x \in L_{\delta_{\alpha+1}} \wedge x \notin L_{\delta_\alpha} \\ & \wedge (\exists M)[M = L_{\delta_{\alpha+1}} \wedge (\forall y \in M)[y \in \mathbb{R} \wedge y <_L x \implies y \in L_{\delta_\alpha}]] \\ \iff & x \in \mathbb{R} \wedge x \in L_{\delta_{\alpha+1}} \wedge x \notin L_{\delta_\alpha} \\ & \wedge (\forall M)(\forall y)[M = L_{\delta_{\alpha+1}} \wedge y \in M \cap \mathbb{R} \wedge y <_L x \implies y \in L_{\delta_\alpha}] \end{aligned}$$

and all notions involved are Δ_1. Now our $D_{\mathcal{C}}$ set A and its complement can be defined over L_{ω_1} by

$$\begin{aligned} x \in A \iff & x \in \mathbb{R} \wedge (\exists \alpha < \omega_1)[\, x - c_\alpha \in L_{\delta_\alpha}\,], \text{ and} \\ x \in \mathbb{R} \setminus A \iff & x \in \mathbb{R} \wedge (\exists \alpha < \omega_1)[\, x \in L_{\delta_{\alpha+1}} \wedge x - c_\alpha \notin L_{\delta_\alpha}\,] \end{aligned}$$

both being Σ_1. Therefore A is Δ_1 definable over L_{ω_1}. This establishes

Theorem 2. *The axiom of constructibility implies the existence of Δ^1_2 sets in $D_{\mathcal{C}}$.* ∎

Sierpiński's propositions ([6, $\mathbf{C}_{68}$–$\mathbf{C}_{70}$]) tell that CH implies sets in $D_{\mathcal{C}}$ with various additional properties. This can also be achieved in the context of Theorem 2. If we choose c_α to be a Cohen real over L_{δ_α} for each $\alpha < \omega_1$ (this may require minor revision of the definition of $\langle \delta_\alpha : \alpha < \omega_1 \rangle$), then the resulting $D_{\mathcal{C}}$ set A becomes a Lusin set (i.e., uncountable set which shares only countably many points with every meager set.) Similarly, if c_α is a random real over L_{δ_α} then A is a Sierpiński set (i.e., uncountable set which shares only countably many points with every null set.) Therefore $V = L$ implies there are two Δ^1_2 sets in $D_{\mathcal{C}}$, one of which is null and lacks the Baire property while the other is meager and non-measurable.

2 Borel Sets

In the previous section we have seen it is consistent that some member of $D_{\mathcal{C}}$ is definable. Here we show that even if some members of $D_{\mathcal{C}}$ are definable, they are not "too simple." That is to say:

Theorem 3. *No Borel sets belong to $D_{\mathcal{C}}$.*

The proof uses the following general fact about $D_{\mathcal{C}}$ sets.

Lemma 2.1. *Let $A \in D_{\mathcal{C}}$. If A has the Baire property, then either A or $\mathbb{R} \setminus A$ is meager. If A is Lebesgue measurable, then either A or $\mathbb{R} \setminus A$ is null.*

Proof: Suppose A has the Baire property, but is not either meager nor comeager, then there are intervals I and J such that $I \setminus A$ and $J \cap A$ are both meager. By cutting down the longer if necessary, we may assume I and J have the same length. Say $I = J + t$. Then since

$$J \setminus ((A + t) \setminus A) = ((I \setminus A) + t) \cup (J \cap A)$$

and this is a meager set, $(A + t) \setminus A$ must be uncountable. The measure version is proved similarly using a pair of intervals on which A and its complement respectively have density greater than $1/2$. ■

The following lemma is of the all importance.

Lemma 2.2. *Let $B \subset \mathbb{R}$ be meager and $P \subset \mathbb{R}$ be perfect. Then there is a number $t \in \mathbb{R}$ such that $(P + t) \setminus B$ contains a perfect set.*

Proof: Let $B \subset \bigcup_{n\in\omega} F_n$ with F_n nowhere dense closed. For each finite binary sequence $\sigma \in {}^{<\omega}2$, we assign $x_\sigma \in P$ by induction on the length $|\sigma|$ of σ. Along with the induction, we also find closed intervals I_n for $n \in \omega$.

First of all, $x_\emptyset$ is an arbitrary member of P and I_0 is an arbitrary non-degenerate closed interval.

Suppose that we have obtained I_n and x_σ for all binary sequence σ of length n. We can take a closed interval I_{n+1} that is contained in the interior of I_n and disjoint from the nowhere dense set $\bigcup_{|\sigma|=n}(F_n - x_\sigma)$. Let us take such I_{n+1} shorter than half of I_n. Then we pick two distinct members $x_{\sigma^\frown 0}$ and $x_{\sigma^\frown 1}$ of P so close to x_σ that for $i \in \{0, 1\}$

$$(I_{n+1} + x_{\sigma^\frown i}) \subset (I_n + x_\sigma) \setminus F_n$$

and

$$|x_{\sigma^\frown i} - x_\sigma| < \frac{1}{2} \min\{\, |x_\tau - x_{\tau'}| : \tau, \tau' \in {}^n2,\ \tau \neq \tau' \,\}.$$

Thus obtained x_σ for all $\sigma \in {}^{<\omega}2$, define $x_\beta = \lim_{n\to\infty} x_{\beta\restriction n}$ for each $\beta \in {}^\omega 2$. We also define t to be the unique common member of all I_n's. Then for any $n \in \omega$ and any $\beta \in {}^\omega 2$, we have $t + x_\beta \in I_{n+1} + x_{\beta\restriction(n+1)} \subset (I_n + x_{\beta\restriction n}) \setminus F_n$. Therefore the set of all reals of the form $t + x_\beta$ is a perfect subset of $P + t$ disjoint from B. ■

Measure version of Lemma 2.2 is also available.

Lemma 2.3. *Let $B \subset \mathbb{R}$ be null and $P \subset \mathbb{R}$ be perfect. Then there is a number $t \in \mathbb{R}$ such that $(P + t) \setminus B$ contains a perfect set.*

Proof: Given a perfect set P, let μ be a Borel probability measure such that $\mu(P) = 1$ and $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Let E be a closed set such that $E \cap B = \emptyset$ and $0 < m(E) < +\infty$ where $m(\ )$ denotes the Lebesgue measure. By invariance of Lebesgue measure, $m(E-x) = m(E)$ for all $x \in \mathbb{R}$. By Fubini's Theorem, we get

$$\begin{aligned} m(E) &= \int_{\mathbb{R}} m(E-x)\, d\mu(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E(t+x)\, dm(t)\, d\mu(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_E(t+x)\, d\mu(x)\, dm(t) \\ &= \int_{\mathbb{R}} \mu(E-t)\, dm(t) \end{aligned}$$

For this is not zero, we have $\mu(E-t) > 0$ for some (positive-measure set of) $t \in \mathbb{R}$. For any such t, we have $\mu(P \cap (E-t)) > 0$ since μ is concentrated on P. Fix one such t. For μ vanishes on countable sets, it follows that $P \cap (E-t)$ is an uncountable closed set. Therefore it contains a perfect subset. So does its translated image $(P+t) \cap E$. Thus we can find a perfect subset of $(P+t) \setminus B$. ■

Now let us prove Theorem 3. Suppose that $B \subset \mathbb{R}$ is a Borel set in $D_{\mathcal{C}}$. By Lemma 2.1, either B or $\mathbb{R} \setminus B$ is meager. For definiteness, say B is meager. If B is uncountable, then it contains a perfect subset P. But then by Lemma 2.2, there is a $t \in \mathbb{R}$ such that $(P+t) \setminus B$ contains a perfect set. Contradicting the assumption that $B \in D_{\mathcal{C}}$. ■

In order to extend Theorem 3 to $\boldsymbol{\Delta}^1_2$ sets, we need the following classic result due to Mansfield and Solovay. Proof can be found in [2] (Section 25, Theorem 25.53 and Lemma 25.54).

Lemma 2.4. (Mansfield and Solovay) *If a (lightface) Σ^1_2 set of reals contains a non-constructible member, then it contains a perfect subset. In other words, every Σ^1_2 set without a perfect subset consists only of constructible reals.* ■

Theorem 4. *Suppose either that every $\boldsymbol{\Delta}^1_2$ set has the Baire property or that every $\boldsymbol{\Delta}^1_2$ set is Lebesgue measurable. Then no $\boldsymbol{\Delta}^1_2$ sets belong to $D_{\mathcal{C}}$.*

Proof: Suppose every $\boldsymbol{\Delta}^1_2$ has the Baire property. Towards the contradiction, let A be a $\boldsymbol{\Delta}^1_2$ set in $D_{\mathcal{C}}$. We may assume A is a (lightface) Δ^1_2 set, since general cases can be handled similarly through straightforward relativization argument. By Lemma 2.1, either A or $\mathbb{R} \setminus A$ is meager. Switching to the complement if necessary, we may assume A is meager. Then by Lemma 2.2, A cannot contain a perfect subset. Therefore Lemma 2.4 implies that $A \subset L$. If $t \in \mathbb{R}$ is not in L, then $(A+t) \cap A = \emptyset$ so $(A+t) \setminus A$ is uncountable. Contradiction. The case of measurability is similar, using Lemma 2.3 instead of Lemma 2.2. ■

Note that our assumption that $\boldsymbol{\Delta}^1_2$ set has the Baire property is strictly weaker than the perfect set theorem for $\boldsymbol{\Delta}^1_2$, from which the result of Theorem 4 follows through same argument as Theorem 3.

Our key lemma 2.2 has first been observed by T. Yorioka using forcing argument, whereas the present combinatorial proof belongs to the present author. Yorioka has pointed out that if every set has the property of Baire and either is countable or contains a perfect subset (just in the case under AD, the axiom of determinacy, or in the Solovay model), then no sets belong to $D_{\mathcal{C}}$. In other words, we have the following

Theorem 5. (Yorioka) *Let κ be a strongly inaccessible cardinal and let $\mathrm{Lv}(\kappa)$ be the Levy partial order that makes all ordinals below κ countable (see [8]). Then in the $\mathrm{Lv}(\kappa)$-generic extension, CH holds and yet no sets in $OD(\mathbb{R})$ belong to $D_{\mathcal{C}}$.*

Therefore, it is consistent (if so is an inaccessible cardinal) that $D_{\mathcal{C}}$ is nonempty but no members of the class are definable.

3 Analytic sets

One question remains open: whether analytic sets in $D_{\mathcal{C}}$ exist. This is difficult because of asymmetry in properties of $\mathbf{\Sigma}^1_1$ and of the dual class $\mathbf{\Pi}^1_1$.

Question. *Is it consistent that there exists an analytic set in $D_{\mathcal{C}}$?*

We do not know the answer yet. Here we record few facts that we know concerning this question.

Theorem 6. *If $A \in \mathbf{\Sigma}^1_1 \cap D_{\mathcal{C}}$, then (1) A is co-meager and co-null and meets every perfect set; and (2) There is a real number r from which every real number is constructible, i.e., $\mathbb{R} \subset L[r]$ holds.*

Proof: (1) follows from Lemmas 2.1 and 2.2 using the perfect set theorem (that every uncountable analytic set contains a perfect subset.) Suppose there is a analytic set A in $D_{\mathcal{C}}$. If A is a $\Sigma^1_1(r)$ set with $r \in \mathbb{R}$. Then, as in our proof of Theorem 4, we have $\mathbb{R} \setminus A \subset L[r]$. As every real number can be expressed as the difference of two members of $\mathbb{R} \setminus A$, $L[r]$ contains all real numbers. This is (2). ■

Readers familiar with A.Miller's [4] might expect that the construction in Theorem 2 could be carried out in uniform Σ_1 way over countable models, so that the set A would be Π^1_1. This is not the case. Indeed, such an argument would produce a pair of Δ^1_1 (hence Borel) sets, contradicting Theorem 3. Moreover, the next theorem seems to impose severe restriction on recursion-theoretic methods, for there is no way to produce a set in $\Pi^1_1 \cap D_{\mathcal{C}}$ by hyperdegree argument.

Theorem 7. *If A is a (lightface) Π^1_1 set which is closed under hyperarithmetical equivalence, then $A \notin D_{\mathcal{C}}$.*

Let us review some recursion-theoretic concepts we need for the proof of Theorem 7.

Let X be a subset of ω. Let ω_1^X be the smallest ordinal $\alpha > \omega$ such that the structure $\langle L_\alpha[X], \in, X\rangle$ is a model of Δ_0-Collection axiom scheme. Despite the name *omega-one*, this is merely a countable ordinal. The smallest such ordinal, namely $\omega_1^\emptyset$ is called *the Church-Kleene omega-one* and denoted by ω_1^{CK}.

Let X and Y be subsets of ω. We say X is *hyperarithmetically reducible* to Y (in symbols, $X \leq_{\mathrm{h}} Y$) if $X \in L_{\omega_1^Y}[Y]$. When $X \leq_{\mathrm{h}} Y$ and $Y \leq_{\mathrm{h}} X$, we say X and Y are *hyperarithmetically equivalent* (in symbols, $X \equiv_{\mathrm{h}} Y$.) The equivalence class of X under $\equiv_{\mathrm{h}}$ is denoted by $[X]_{\mathrm{h}}$ and called the *hyperdegree* of X. The hyperdegrees are partially ordered by $\leq_{\mathrm{h}}$. The smallest hyperdegree, denoted by $\mathbf{0}$, is that of Δ^1_1 definable subsets of ω, namely members of $\mathcal{P}(\omega) \cap L_{\omega_1^{\mathrm{CK}}}$. Note that for each $Y \subset \omega$ there are only countably many X which is hyperarithmetically reducible to Y. In particular, each $[X]_{\mathrm{h}}$ consists of countably many subsets of ω.

These definitions extends naturally to other countable objects such as real numbers, infinite sequences of integers, etc.

A real number x is said to be *quickly constructible* if $x \in L_{\omega_1^x}$. Let C_1 be the set of all quickly constructible reals. The next lemma lists some basic facts about C_1.

Lemma 3.1. *(1) C_1 is a Π^1_1 set. (2) A Π^1_1 set A contains a perfect set if and only if $A \not\subset C_1$. (3) C_1 is contained in L. (4) Every constructible real is hyperarithmetically reducible to some member of C_1. (5) Hyperdegrees of members of C_1 is wellordered by $\leq_{\mathrm{h}}$ into the order-type ω_1^L.* ■

The essential part of our proof of Theorem 7 is formulated as follows.

There is a perfect binary tree $T \subset {}^{<\omega}2$ of which every path is a Cohen real over $L_{\omega_1^{\mathrm{CK}}}$. In fact, there is such a T whose hyperdegree is as low as, $\mathbf{0}'$, the second smallest hyperdegree in C_1. Fix such T. Let $\langle s_n : n \in \omega\rangle \in L_{\omega_1^{\mathrm{CK}}}$ be an enumeration of ${}^{<\omega}2$. Define a real number $b = 0.b_1b_2b_3\cdots$ (decimal) by

$$b_{2n+1} = \begin{cases} 2, & (s_n \in T); \\ 3, & (s_n \notin T); \end{cases}$$
$$\text{and} \quad b_{2n+2} = 0.$$

Clearly, b and T are hyperarithmetically equivalent.

The perfect tree T naturally codes a homeomorphism of ${}^\omega 2$ onto the set of all paths of T. For each $\beta \in {}^\omega 2$ let $T(\beta)$ be the path of T that corresponds to β by the natural homeomorphism. Then we have $T(\beta) \leq_{\mathrm{h}} \langle T, \beta\rangle$ and $\beta \leq_{\mathrm{h}} \langle T(\beta), T\rangle$.

Given $\beta \in {}^\omega 2$, define a real number $a = 0.a_1a_2a_3\cdots$ (decimal) by

$$a_{2n+1} = 0;$$
$$\text{and} \quad a_{2n+2} = 4 + T(\beta)(n).$$

Then a and $T(\beta)$ are hyperarithmetically equivalent.

Finally let $c = a + b$. Then $c \equiv_{\mathrm{h}} \langle T, T(\beta) \rangle$. Therefore if $\beta \geq_{\mathrm{h}} T$, then c and β are hyperarithmetically equivalent.

Now, $a = c - b$ is hyperarithmetically equivalent to $T(\beta)$ which is a Cohen real over $L_{\omega_1^{\mathrm{CK}}}$. By genericity, we have $T(\beta) \notin L_{\omega_1^{\mathrm{CK}}}$ and $\omega_1^{T(\beta)} = \omega_1^{\mathrm{CK}}$. Therefore a is not a quickly constructible real.

Lemma 3.2. *Let A be an uncountable subset of C_1 which is closed under hyperarithmetical equivalence. Then $A \setminus (A + b)$ is uncountable.*

Proof: Let x be a member of A such that $T \leq_{\mathrm{h}} x$. By the assumption $[x]_{\mathrm{h}} \subset A$. Define a and c from any $\beta \in {}^{\omega}2$ such that $\beta \equiv_{\mathrm{h}} x$. Then $c \equiv_{\mathrm{h}} x$ so $c \in A$. Since $c - b = a \notin C_1$, we have $c \in A \setminus (A + b)$. From this it follows that $A \setminus (A + b)$ meets hyperdegree of every $x \in A$ such that $x \geq_{\mathrm{h}} T$. But by results of Lemma 3.1, there are uncountably many such hyperdegrees in A. ■

Theorem 7 follows immediately from Theorem 6 and Lemmas 3.1–3.2.

Acknowledgement. I would like to thank Sakaé Fuchino, Daisuke Ikegami, Masaru Kada and Yasuo Yoshinobu for inspiring discussion and kind advices. Special thanks go to Teruyuki Yorioka who introduced me to the problem.

References

[1] S.Banach, *Sur les transformations biunivoques*, Fund. Math., 19 (1932), 10–16.

[2] T.Jech, *Set Theory — the 3rd Millennium Edition*, Springer.

[3] M.Laczkovich, *Two constructions of Sierpiński and some cardinal invariants of ideals*, Real Analysis Exchange **24**–2 (1998), 663–676.

[4] A.W.Miller, *Infinite combinatorics and definability*, Ann. Pure Appl. Logic **41** (1989), 179–203.

[5] J.Shinoda, *Some consequences of Martin's axiom and the negation of the continuum hypothesis*, Nagoya Math. J. **49** (1973), 117–125.

[6] W.Sierpiński, *Hypotèse du Continu*, Chelsea Publ.

[7] W.Sierpiński, *Sur les translations des ensembles des linéaires*, Fund. Math. **19** (1932), 22–28.

[8] R.M.Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math., (2nd ser.) **92** (1970), 1–56.