Equivalence of Lebesgue measurability with determinacy of covering games*

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March 23, 2006

Let $\mathcal{C}$ be the Cantor space, i.e., the set $\{0,1\}^\omega$ topologized with the product topology, taking $\{0,1\}$ discrete. Let $\{J_n\}_{n\in\omega}$ enumerate in a straightforward way the basic clopen sets in $\mathcal{C}$. Let $\{G(k)\}_{k\in\omega}$ recursively enumerate all the finite unions of $J_n's$.

We give the standard product measure on $\mathcal{C}$. In what follows, this product measure is called *the Lebesgue measure.* This abuse of language would cause little confusion, since the Cantor space $\mathcal{C}$ and the unit interval $[0,1]$ are measure theoretically very similar. In fact, removing an appropriate countable set (i.e., the sequences of 0's and 1's having only finitely many places for 1) from $\mathcal{C}$, we obtain a measure space which is isomorphic to $[0,1]$. For this reason, we also call elements of $\mathcal{C}$ *reals.*

Let us denote by m, m_* and m^* respectively, the Lebesgue measure, its inner and outer extensions respectively. We may assume that the enumerations $\{J_n\}_{n\in\omega}$ and $\{G(k)\}_{k\in\omega}$ have been made so that the relations $m(J_n) < p/(q+1)$, $m(G(k)) < p/(q+1)$ and similar relations with "$>$" replacing "$<$" are all recursive (for n, k, p, q in ω).

Covering games have been introduced by L. Harrington in order to give a simpler proof of a theorem of J. Mycielski and S. Swierczkowski ([1]) that the axiom of determinacy implies every set of reals is Lebesgue measurable.

Let $A \subset \mathcal{C}$. Given a rational number $\varepsilon > 0$, we consider the following two-person infinite game:

$$\begin{array}{lccccccc}
\text{(I)} & a_0 & & & a_1 & & & \cdots \\
 & & \searrow & \nearrow & & \searrow & \nearrow & \\
\text{(II)} & & k_0 & & & k_1 & & \cdots
\end{array}$$

where $a_i \in \{0,1\}$ and $k_i \in \omega$. We impose the following restriction on Player II's choices: k_i must satisfy $m(G(k_i)) < \varepsilon/4^i$ for all $i \in \omega$. A course of choices of Player I specifies a real

$$\alpha = (a_0, a_1, \ldots, a_i, \ldots)$$

*This article is a part of my Doctor's Thesis submitted to Nagoya University on March 2001.

while Player II specifies an open subset of G of $\mathcal{C}$:

$$G = G_0 \cup G_1 \cup \cdots \cup G_i \cup \cdots .$$

Player I wins if $\alpha \in A \setminus G$. Otherwise Player II wins. We call this game the covering game associated with A and ε and denote it by $\mathcal{G}(A : \varepsilon)$. This game is closely related to Lebesgue measurability.

Lemma 0.1 *Let $A \subset \mathcal{C}$. Let $\varepsilon > 0$ be rational. Consider the game $\mathcal{G}(A : \varepsilon)$.*

(1) *If Player I has a winning strategy, then $m_*(A) \geq \varepsilon$;*

(2) *If Player II has a winning strategy, then $m^*(A) < 4\varepsilon$.*

Proof: (1) Suppose that Player I has a winning strategy σ in $\mathcal{G}(A : \varepsilon)$. Let S be the set of courses of legal moves of Player II:

$$S = \{\, \gamma \mid (\forall i)[\, m(G(\gamma(i)) < \varepsilon/4^i \,] \,\}.$$

Let H be the set of reals which Player I specifies by playing according to σ which Player II plays legally:

$$H = \{\, \alpha \in \mathcal{C} \mid (\exists \gamma \in S)(\forall i)[\, \alpha(i)\rangle = \sigma(\gamma(0), \ldots, \gamma(i-1))\,] \,\}.$$

It is easy to see that S is closed, hence H is $\mathbf{\Sigma}^1_1$. For σ is a winning strategy of Player I, we have $H \subset A$. Being $\mathbf{\Sigma}^1_1$, H is Lebesgue measurable. Therefore, in order to prove $m_*(A) \geq \varepsilon$, it is sufficient to show $m(H) \geq \varepsilon$.

Suppose contrary that $m(H) < \varepsilon$. Then there exists a sequence $\{n_p\}_{p \in \omega}$ of integers such that

$$H \subset \bigcup_{p \in \omega} J_{n_p} \quad \text{and} \quad \sum_{p \in \omega} m(J_{n_p}) < \varepsilon.$$

For each $i \in \omega$ let u_i be the smallest integer u such that

$$\sum_{p \geq u} m(J_{n_p}) < \frac{\varepsilon}{8^i}.$$

Let $\gamma(i) = k_i$ be an index of the finite union

$$G(k_i) = \bigcup \{\, J_{n_p} \mid u_i \leq p < u_{i+1} \,\}.$$

Then γ is a course of legal choices of Player II in $\mathcal{G}(A : \varepsilon)$ which defeats σ. Contradiction.

(2) Suppose that τ is a winning strategy of Player II in $\mathcal{G}(A : \varepsilon)$. Let D be the union of all open sets $G(k)$ which τ tells Player II to choose against Player I's choices:

$$D = \bigcup \{\, G(\tau(a_0 \ldots, a_i)) \mid a_0, \ldots, a_i \in \{0,1\},\ i \in \omega \,\}.$$

Straightforward computation shows $m(D) < 4\varepsilon$. Since τ is winning of Player II, we have $A \subset D$. (Q.E.D)

In fact, determinacy of covering games is, in a certain sense, equivalent to Lebesgue measurability. We will return to this aspect later. As a consequence of Lemma 0.1, we obtain a lightface version of the result of Mycielski and Swierczkowski.

Lemma 0.2 *Let Γ be an adequate pointclass containing Π^0_1. Suppose that the game $\mathcal{G}(A : \varepsilon)$ is determined for every $A \subset \mathcal{C}$ in Γ and every rational $\varepsilon > 0$. Then every Γ-set in $\mathcal{C}$ is Lebesgue measurable.*

PROOF: Suppose that a Lebesgue non-measurable set $A \subset \mathcal{C}$ belonging to $\exists^{\mathcal{C}}\Gamma$ exists. Let B_i and B_o be Borel sets such that $B_i \subset P \subset B_o$, $m(B_i) = m_*(A)$ and $m(B_o) = m^*(A)$. Then $m(B_o \setminus B_i) > 0$. By the Lebesgue Density Lemma, there exists a basic clopen set J_n such that

$$m(J_n \cap (B_o \setminus B_i)) > \frac{8}{9} m(J_n).$$

From this it follows that

$$m_*(J_n \cap A) < \frac{1}{5} m(J_n) \quad \text{and} \quad m^*(J_n \cap A) > \frac{4}{5} m(J_n).$$

Here we may assume without loss of generality that J_n is of the form $\{ \alpha \mid s \subset \alpha \}$ for some finite binary sequence $s \in \{0,1\}^{<\omega}$. Let $A' = \{ \alpha \mid s^\frown\alpha \in A \}$. Then A' belongs to Γ since this pointclass is closed under taking preimages via recursive mappings. By the inequalities above, we have $m_*(A') < 1/5$ and $m^*(A') > 4/5$. Then by Lemma 0.1, neither player has a winning strategy in $\mathcal{G}(A : 1/5)$. (Q.E.D)

Now let **LM** denote the statement *"every set of reals is Lebesgue measurable."* and **ADC** denote *"all covering games are determined."* What Harrington has proved is that **ADC** implies **LM**. We show the converse of this, hence equivalence of **LM** and **ADC**.

Theorem 1 *Let $A \subset \mathcal{C}$ be a Lebesgue measurable set. Then for every positive number ε, the covering game $\mathcal{G}(A : \varepsilon)$ is determined.*

PROOF: Let $H \subset A$ be a Borel set such that $m(H) = m(A)$. Let us consider another covering game $\mathcal{G}(H : \varepsilon)$. In fact, we can find such H among $\mathbf{\Sigma^0_2}$ sets. This game is determined since the winning condition is Borel. We show that the player who has a winning strategy for $\mathcal{G}(H : \varepsilon)$ wins $\mathcal{G}(A : \varepsilon)$.

If Player I has a winning strategy for $\mathcal{G}(H : \varepsilon)$, then the same player easily wins $\mathcal{G}(A : \varepsilon)$ by using the same strategy, since $H \subset A$.

Suppose on the other hand that Player II has a winning strategy for the game $\mathcal{G}(H : \varepsilon)$. Let τ be one such winning strategy. Then for any finite sequence $(a_0, \ldots, a_i)$ of zeros and ones we have

$$m(G(\tau(a_0, \ldots, a_i))) < \frac{\varepsilon}{4^{i+1}}.$$

For each $i \in \omega$ define δ_i by

$$\delta_i = \frac{\varepsilon}{4^{i+1}} - \max\{\, m(G(\tau(a_0, \dots, a_i))) \mid a_0, \dots, a_i \in \{0,1\}\,\}.$$

Then δ_i are positive for all $i \in \omega$.

Since A is measurable, $A \setminus H$ is a null set. Therefore it can be covered by a countable family $\{N(s_n)\}_{n\in\omega}$ of basic clopen sets of which the sum of volumes is less than δ_0:

$$A \setminus H \subset \bigcup_{n\in\omega} N(s_n) \quad \text{and} \quad \sum_{n\in\omega} m(N(s_n)) < \delta_0.$$

Find a strictly increasing sequence $\{n_i\}_{i\in\omega}$ of integers such that for each $i \in \omega$

$$\bigcup_{n\in\omega} N(s_n) \quad \text{and} \quad \sum_{n_i \le n \in\omega} m(N(s_n)) < \delta_i.$$

In the game $\mathcal{G}(A : \varepsilon)$ let Player II play, against Player I's moves $a_0, \dots, a_i$, the integer k_i such that

$$G(k_i) = G(\tau(a_0, \dots, a_i)) \cup \bigcup_{n_i \le n < n_{i+1}} N(s_n).$$

We show that this gives a winning strategy of Player II for $\mathcal{G}(A : \varepsilon)$. Let Player II play by this strategy, producing k_i ($i = 0, 1, 2, \dots$) against Player I's $\alpha = (a_0, a_1, a_2, \dots)$. The moves are legal, because

$$\begin{aligned} m(G(k_i)) &\le m(\tau(a_0, \dots, a_i)) + m\left(\bigcup_{n_i \le n < n_{i+1}} N(s_n)\right) \\ &< m(\tau(a_0, \dots, a_i)) + \delta_i \\ &\le \frac{\varepsilon}{4^{i+1}}. \end{aligned}$$

If $\alpha \notin A$ then Player II wins by definition. If $\alpha \in A$ then either $\alpha \in H$ or $\alpha \in A \setminus H$. Corresponding to each case, we have $\alpha \in G(\tau(a_0, \dots, a_i))$ for some $i \in \omega$ (since τ is Player II's winning strategy for $\mathcal{G}(H : \varepsilon)$) or $\alpha \in \bigcup_{n_i \le n < n_{i+1}} N(s_n)$ for some $i \in \omega$ (since $\{N(s_n)\}_{n\in\omega}$ covers $A \setminus H$). Therefore we have anyway $\alpha \in G(k_i)$ for some $i \in \omega$. Therefore this strategy is winning. (QED)

Therefore **ADC** and **LM** are equivalent statements on the basis of **ZF**+**DC**. This fact suggests that the use of covering games for deriving measurability from determinacy is indeed a right way, because the result (measurability) tells that the tool (determinacy of covering game) is necessary.

References

[1] Jan Mycielski and S. Świerczkowski. On the Lebesgue measurability and the axiom of determinateness. *Fundamenta Mathematicae*, 54:67–71, 1964.