On points of differentiability of discontinuous functions

Hiroshi Fujita* (藤田 博司)
Ehime University

January 21, 2011

Metric theory of Diophantine approximations tells that *almost every* irrational number α (in the sense of Lebesgue measure) has the following property: *there exists a positive constant* $c(\alpha)$*, depending only on* α*, such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c(\alpha)}{q^2}$$

for every rational number p/q*.* It follows that if you define a real function $f : \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(x) = \begin{cases} 1/q^3, & \text{if } x = p/q, \\ 0, & \text{if } x \text{ is irrational}, \end{cases}$$

then f is discontinuous at every rational numbers, continuous at every irrational numbers and differentiable almost everywhere. By another well-known theorem due to Dirichlet, if you instead put

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q, \\ 0, & \text{if } x \text{ is irrational}, \end{cases}$$

then f is still continuous at every irrational number, but now it is nowhere differentiable.

In general, *how are the points of differentiability of a real function distributed on the real line?* Concerning this general question, we prove the following

---

*The author would like to dedicate this small article to Professor Hiroshi Sakai, the organizer of this RIMS conference, on the special occasion of his marriage. *Happy Wedding!*

Theorem 1 *Let A and B be disjoint F_σ sets of real numbers. There exists a real function $f : \boldsymbol{R} \to \boldsymbol{R}$ which is (1) discontinuous at every $x \in A$, (2) continuous anywhere else, and (3) differentiable at every $x \in B$.*

Proof. Let $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ be increasing sequences of compact sets so that

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots,$$
$$B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots,$$
$$A = \bigcup_{n=1}^\infty A_n, \quad \text{and} \quad B = \bigcup_{n=1}^\infty B_n.$$

Being a disjoint pair of compact sets, A_n and B_n satisfy condition

$$\min\left\{ |a-b| \;\middle|\; a \in A_n \text{ and } b \in B_n \right\} > 0.$$

Therefore we can choose a sequence $\{r_n\}_{n=1}^\infty$ of positive numbers so that

$$a \in A_n \wedge b \in B_n \implies |a-b| \geq r_n, \text{ and}$$
$$r_n \searrow 0 \quad (\text{as } n \to \infty.)$$

Choose another sequence $\{s_k\}_{k=1}^\infty$ of positive numbers so small that

$$\frac{1}{r_n} \sum_{k=n}^\infty s_k \to 0 \quad (\text{as } n \to \infty.)$$

Finally put

$$f(x) = \sum_{k=1}^\infty s_k \cdot \chi_{A'_k}(x)$$

where $\chi_{A'_k}$ is the characteristic functions of a countable dense subset A'_k of A_k.

Suppose $a \in A$. Say $a \in A_n$. Then arbitrarily close to a exist points $x \in A'_k$ and $y \notin \bigcup A'_n$, (note that one of x and y may be identical with a,) so that $f(x) \geq s_k$ and $f(y) = 0$. It follows that f is discontinuous at a. This proves (1).

Now suppose $a \notin A$. We show that f is continuous at a. Given $\varepsilon > 0$, choose n so large that $\sum_{k>n} s_k < \varepsilon$. Since $a \notin A_n$ and A_n is closed, there exists a positive number δ such that

$$(a-\delta, a+\delta) \cap A_n = \emptyset.$$

If $|x - a| < \delta$ then $x \notin A_n$, and then

$$|f(x) - f(a)| = f(x) \leq \sum_{k>n} s_k < \varepsilon.$$

Therefore f is continuous at a. (2) is thus verified.

Let $b \in B$. Consider a variable x moving towards b. If x is so close to b that $|x-b| < r_1$, there exists a unique number i such that $r_{i+1} \leq |x-b| < r_i$. Then $x \notin A_i$ and we have

$$f(x) \leq \sum_{k=i+1}^{\infty} s_k.$$

On the other hand, we have $f(b) = 0$ and $|x - b| \geq r_{i+1}$. It follows that

$$\left| \frac{f(x) - f(b)}{x - b} \right| \leq \frac{1}{r_{i+1}} \sum_{k=i+1}^{\infty} s_k.$$

The right hand side goes to zero as $i \to \infty$. But as x gets closer and closer to b, the unique i such that $r_{i+1} \leq |x - b| < r_i$ must go big indefinitely. From this it follows that f is differentiable at b with coefficient zero. This proves (3). $\square$

The theorem tells that the distribution of points of discontinuity and of points of differentiability is rather arbitrary. For example, a real function can be discontinuous almost everywhere and yet differentiable at points which are uncountably dense in every interval. Another real function can be differentiable almost everywhere and discontinuous on an uncountably dense set. However, this freedom of choice is not complete. For example, if a real function is discontinuous at every rational number, it cannot be differentiable at all irrational numbers. In fact, *if the set of points of discontinuity of f is dense, then the set of points of differentiability is meager* (i.e., of first category in the sense of Baire). To see this, it is sufficient to prove the following

Theorem 2 *For every real function $f : \boldsymbol{R} \to \boldsymbol{R}$ there exists an F_σ set E such that (1) if f is differentiable at α then $\alpha \in E$, (2) f is continuous at every $\alpha \in E$.*

Proof. Suppose that f is differentiable at α. We have, by definition,

$$\forall\varepsilon > 0 \exists\delta > 0 \forall x \left(0 < |x-\alpha| < \delta \implies \left| \frac{f(x)-f(\alpha)}{x-\alpha} - f'(\alpha) \right| < \varepsilon \right).$$

Putting $\varepsilon = 1$, we then obtain

$$\forall x \Big(|x-\alpha| < \delta \implies |f(x)-f(\alpha)| \le (|f'(\alpha)|+1)\cdot|x-\alpha| \Big)$$

for every *sufficiently small* $\delta > 0$. There is a positive number M (just let $M \ge 2(|f'(\alpha)|+1)$,) such that for every sufficiently small $\delta > 0$,

$$\forall x \forall y \big(|x-\alpha| < \delta \wedge |y-\alpha| < \delta \implies |f(x)-f(y)| \le M\cdot\delta \big),$$

in other words,

$$\forall x \forall y \big(|x-\alpha| \ge \delta \vee |y-\alpha| \ge \delta \vee |f(x)-f(y)| \le M\cdot\delta \big).$$

It is easy to see that the last condition defines a closed set of α for each fixed M and δ. Let $F(M,\delta)$ be this closed set. Then put

$$E = \bigcup_{M=1}^{\infty} \bigcup_{n=1}^{\infty} \left(\bigcap \left\{ F(M,\delta) \,\middle|\, 0 < \delta < \frac{1}{n} \right\} \right).$$

This defines an F_σ set. While every point of differentiability of f belongs to E, it follows from the definition of $F(M,\delta)$ that f is continuous at every $\alpha \in E$. $\square$

Author's E-mail address: `fujita.hiroshi.mh@ehime-u.ac.jp`