On effectivization of Freiling's Axioms of Symmetry

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Freiling's *Axiom of Symmetry* ($A_{\aleph_0}$) is the following statement: *For every function $F : 2^\omega \to [2^\omega]^{\leq\omega}$ which assigns a countable set of reals to each real, there exist two distinct reals, say a and b, such that $a \notin F(b)$ and $b \notin F(a)$.*

Fact 1 (Freiling[1]). ZFC $\vdash A_{\aleph_0} \leftrightarrow \neg$CH. $\triangleleft$

Galen Weitkamp has considered (in [3]) an effective version of $A_{\aleph_0}$.

Fix a recursive bijection $\langle\ ,\ \rangle : \omega \times \omega \to \omega$. For each $a \in 2^\omega$ and $n \in \omega$, the real $(a)_n \in 2^\omega$ is defined by $(a)_n(k) = a(\langle n, k\rangle)$. In this way every real $a \in 2^\omega$ naturally codes a countable set $\{\, (a)_n : n \in \omega \,\}$.

Definition. Let Γ be a pointclass. Then $A(\Gamma)$ states: *Let $f : 2^\omega \to 2^\omega$ be a function whose graph as subset of $2^\omega \times 2^\omega$ belongs to the class Γ, then there exist two distinct reals a and b such that*

$$\forall n \in \omega \Big[x \neq \big(f(y)\big)_n \;\&\; y \neq \big(f(x)\big)_n \Big].$$

Fact 2 (Weitkamp [3]).

(1) ZF + DC $\vdash A(\mathbf{\Sigma}^1_1)$.

(2) $A(\Pi^1_1) \leftrightarrow A(\Sigma^1_2) \leftrightarrow 2^\omega \not\subset L$. $\triangleleft$

Fact 2 (2) gives an effective version of Freiling's Fact 1. However, there are some difficulties within Weitkamp's formulation:

1. Freiling has considered A_{null} and A_{meager} as well, replacing "countable" by "null" and "meager" respectively. It is not clear how we can modify Weitkamp's setting to handle these generalizations.

2. Giving a countable set of reals is not the same thing as giving its code. From a code you can easily obtain a countable set as Weitkamp does. But for each countable set $C \in [2^\omega]^{\leq\omega}$ there exist uncountably many reals which codes C, and you do not know how to choose one.

To investigate this second point more closely, suppose we are given a relation $R \subset 2^\omega \times 2^\omega$ which is somehow *nicely definable* (Borel, analytic, or anything). Suppose also that for every $x \in 2^\omega$ the vertical section $R_x = \{ y : R(x,y) \}$ is nonempty and countable. In such a case can you always *define* a function $f : 2^\omega \to 2^\omega$ such that $R_x = \{ (f(x))_n : n \in \omega \}$? For example, the following question should be a challenging exercise:

Question 3. *Define a function $f : 2^\omega \to 2^\omega$ so that*

$$\Big\{ \big(f(x)\big)_n : n \in \omega \Big\} = \Big\{ y \in 2^\omega : y \textit{ is recursive in } x \Big\}$$

for every $x \in 2^\omega$. At which level of the arithmetical hierarchy can such f be?

From this point of view, the following reformulation seems more natural to me.

Definition. Let $A^*(\Gamma)$ state: *For a relation $R \subset 2^\omega \times 2^\omega$ in Γ, if every vertical section R_x is countable, then there are two distinct reals a and b such that both $R(a,b)$ and $R(b,a)$ fail.*

This is not always equivalent to Weitkamp's $A(\Gamma)$. We still have

$$A^*(\Sigma^1_2) \leftrightarrow A^*(\Delta^1_2) \leftrightarrow 2^\omega \not\subset L,$$

so $A^*(\Sigma^1_2)$ and $A(\Sigma^1_2)$ are equivalent. On the other hand, we have (by the Fubini Theorem)

$$\mathrm{ZF} + \mathrm{DC} \vdash A^*(\mathbf{\Pi}^1_1).$$

Therefore $A^*(\Pi^1_1)$ is strictly weaker than $A(\Pi^1_1)$.

Our version has one obvious advantage. It is quite easy to formulate $A^*_{\mathrm{null}}(\Gamma)$ and $A^*_{\mathrm{meager}}(\Gamma)$. Then by Fubini and Kuratowski-Ulam Theorems,

Fact 4. *For every pointclass Γ,*

(1) $\mathbf{LM}(\Gamma) \to A^*_{\mathrm{null}}(\Gamma)$, and

(2) $\mathbf{BP}(\Gamma) \to A^*_{\mathrm{meager}}(\Gamma)$. $\triangleleft$

It is amusing to point out that in certain cases these arrows are inverted.

Fact 5.

(1) $\mathbf{LM}(\Delta^1_2) \leftrightarrow A^*_{\mathrm{null}}(\Delta^1_2)$, and

(2) $\mathbf{BP}(\Delta^1_2) \leftrightarrow A^*_{\mathrm{meager}}(\Delta^1_2)$.

Here, I will give only a proof of (1), since (2) can be proved similarly.

We already know that $\mathbf{LM}(\Delta^1_2)$ implies $A^*_{\mathrm{null}}(\Delta^1_2)$. To see the converse, suppose that $\mathbf{LM}(\Delta^1_2)$ *fails.* Then there is no random real over L. In other words, every real $r \in 2^\omega$ belongs to some null G_δ set with constructible code.

Let $U \subset 2^\omega \times 2^\omega$ be a universal G_δ set which is lightface Π^0_2. Then our hypothesis $\neg\mathbf{LM}(\Delta^1_2)$ can be written as

$$\forall r \in 2^\omega \exists c \in 2^\omega \Big[c \in L \;\&\; \mu(U_c) = 0 \;\&\; r \in U_c \Big].$$

where μ denotes the Lebesgue measure. Since the $\big[\ldots\big]$ part of the statement is Σ^1_2, the Novikov-Kondô-Addison Theorem gives a Δ^1_2 function $\varphi : 2^\omega \to 2^\omega$ such that

$$\forall r \in 2^\omega \Big[\varphi(r) \in L \;\&\; \mu(U_{\varphi(r)}) = 0 \;\&\; r \in U_{\varphi(r)} \Big].$$

Let $<^*$ be a Σ^1_2 wellordering of $2^\omega \cap L$ into order-type ω_1. We may assume

$$L \models \big[<^* \text{ is a } \Sigma^1_2\text{-good wellordering} \big]$$

in the sense explained in Section 5A of [2]. Now define $R \subset 2^\omega \times 2^\omega$ by

$$R(x, y) \iff \exists c \leq^* \varphi(x) \Big[\mu(U_c) = 0 \;\&\; y \in U_c \Big].$$

It is straightforward to see that every vertical section R_x is null and that every two reals a and b satisfy either $R(a, b)$ or $R(b, a)$ according to $\varphi(b) \leq^* \varphi(a)$ or not. Thus what remains to see is:

Lemma 6. *The relation R is Δ^1_2.*

PROOF. Let $\mathrm{IS}(x, y)$ be the predicate that tells x codes the initial segment of $\leq^*$ with top y. Exercise 5A.1 of [2] shows that $V = L$ implies that IS is Δ^1_2. Even when $V \neq L$, the predicate

$$\mathrm{IS}'(x, y) \iff x, y \in 2^\omega \cap L \;\&\; L \models \mathrm{IS}(x, y)$$

is still Σ^1_2. We then have

$$\begin{aligned} \neg R(x, y) &\leftrightarrow \forall c \leq^* \varphi(x) \Big[\mu(U_c) > 0 \vee y \notin U_c \Big] \\ &\leftrightarrow \exists b \Big[b \in L \;\&\; \mathrm{IS}'(b, \varphi(x)) \;\&\; \forall n \in \omega \big[\mu(U_{(b)_n}) > 0 \vee y \notin U_{(b)_n} \big] \Big] \end{aligned}$$

which gives a Σ^1_2 description of negation of R. ◁

This completes the proof of Fact 5.

Question 7. *Does $A^*_{\mathrm{null}}(\Sigma^1_2)$ imply $\mathbf{LM}(\Sigma^1_2)$?*

References

[1] Ch.Freiling, *Axiom of Symmetry, Throwing Darts at the Real Number Line*, Jour. Symb. Logic, **51** (1986), pp.190–200.

[2] Y.N.Moschovakis, **Descriptive Set Theory** (2nd Edition), American Mathematical Society 2009.

[3] G.Weitkamp, *The* Σ^1_2 *theory of axioms of symmetry*, Jour. Symb. Logic, **54** (1989), pp.727–734.